Economical Extremal Hypergraphs for the Erdős–Selfridge Theorem

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1. Abstract

A positional game can be thought of as a generalization of Tic-Tac-Toe played on a hypergraph $(V, \mathcal{H})$. We study the Maker-Breaker game in which Maker wins if she occupies all of the vertices in an edge of $\mathcal{H}$; otherwise Breaker wins. The Erdős–Selfridge Theorem, a significant result in positional game theory, gives criteria for the existence of an explicit winning strategy for Breaker for the game played on $\mathcal{H}$. The bound in this theorem has been shown to be tight, as there are several examples of extremal hypergraphs for this theorem. We focus on the n-uniform extremal hypergraphs on which Maker has an economical (n-turn) winning strategy. We prove two distinct characterizations of these economical extremal hypergraphs.

2. Background

A **positional game** played on a hypergraph $(V, \mathcal{H})$ is a two-player game in which the players alternately occupy open vertices from V.

In a **strong positional game**, the first player to occupy every vertex in some edge A in $\mathcal{H}$ wins the game.

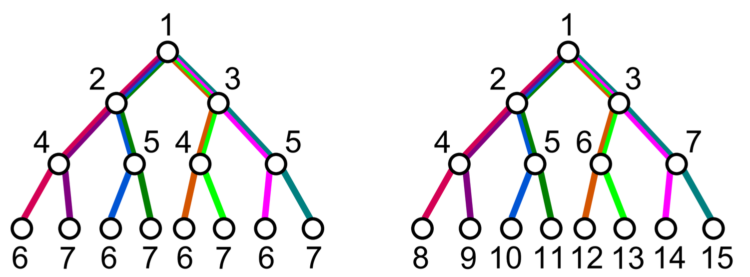
In a **Maker-Breaker game** played on $\mathcal{H}$, the first player, **Maker**, wins if she occupies every vertex in an edge of $\mathcal{H}$; otherwise, the second player, **Breaker**, wins.

Erdős–Selfridge Theorem:

◇ If $\sum_{A \in \mathcal{H}} 2^{-|A|} < \frac{1}{2}$, then Breaker has an explicit winning strategy for the Maker–Breaker game played on $\mathcal{H}$.

◇ If $\mathcal{H}$ is n-uniform, this bound simplifies to $|\mathcal{H}| < 2^{n-1}$.

An **extremal hypergraph** for the Erdős–Selfridge Theorem is an n-uniform hypergraph with 2^{n-1} edges on which Maker has a winning strategy.



Two examples of 4-uniform extremal hypergraphs for the Erdős–Selfridge Theorem: the original example given by Erdős and Selfridge [2] and an example given by Beck [1].

In the examples above, the **vertices** are the labels on the nodes of the binary tree, and the **edges** are the sets of labels on the root-to-leaf paths of the tree.

Both are 4-uniform hypergraphs with 8 edges on which Maker has a winning strategy: First, she occupies the root vertex 1, and then she follows a **sibling-pairing strategy** for her subsequent turns, choosing the sibling of whichever vertex Breaker had occupied in the previous turn.

Let T be a complete n-level binary tree.

A **good labeling** of T is a labeling $L : V(T) \to \mathbb{N}$ such that:

1. If N and N' are siblings, then $L(N) \neq L(N')$;
2. If $N_i < N_j$, then $L(N_i) \neq L(N_j)$;
3. If N_i and N_i' are siblings and N_j and N_j' are siblings, then $L(N_i) = L(N_j)$ (if and) only if $L(N_i') = L(N_j')$.

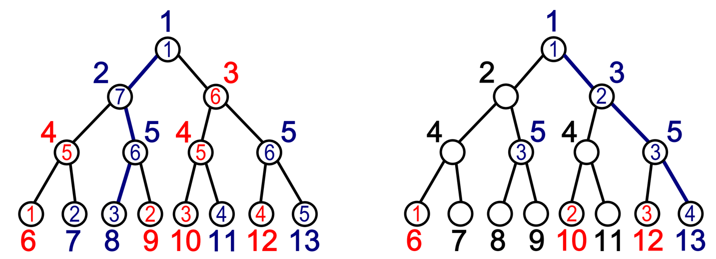
A hypergraph $\mathcal{H}$ is **realized** by a labeling L of T if the vertices of $\mathcal{H}$ are the labels on the nodes of T and the edges of $\mathcal{H}$ are the sets of labels on the root-to-leaf paths of T, i.e., $N_1, N_2, \ldots, N_n$ are the nodes of a root-to-leaf path in T if and only if $\{L(N_1), L(N_2), \ldots, L(N_n)\}$ is an edge in $\mathcal{H}$.

A **tumbleweed** is a hypergraph realized by a good labeling of T.

Tumbleweeds are extremal hypergraphs, since Maker can follow her sibling-pairing strategy to win the game played on any tumbleweed.

Breaker can take advantage of Maker's sibling-pairing strategy to force Maker to win the game as slowly as possible (as in the example below on the left).

We are interested in the hypergraphs on which Maker has an **economical** (n-turn) winning strategy (as in the example below on the right).

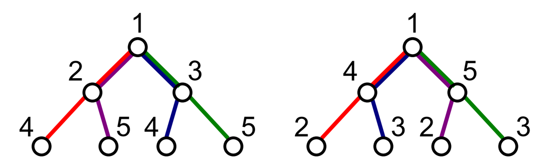


An **economical extremal hypergraph** for the Erdős–Selfridge Theorem is an n-uniform hypergraph with 2^{n-1} edges on which Maker has an economical winning strategy.

Question: Characterize the economical extremal hypergraphs for the Erdős–Selfridge Theorem.

3. First Characterization

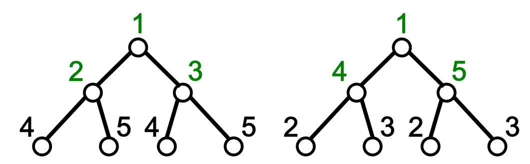
The same tumbleweed can be realized by different labelings of the binary tree.



Two labelings which realize the same 3-uniform tumbleweed.

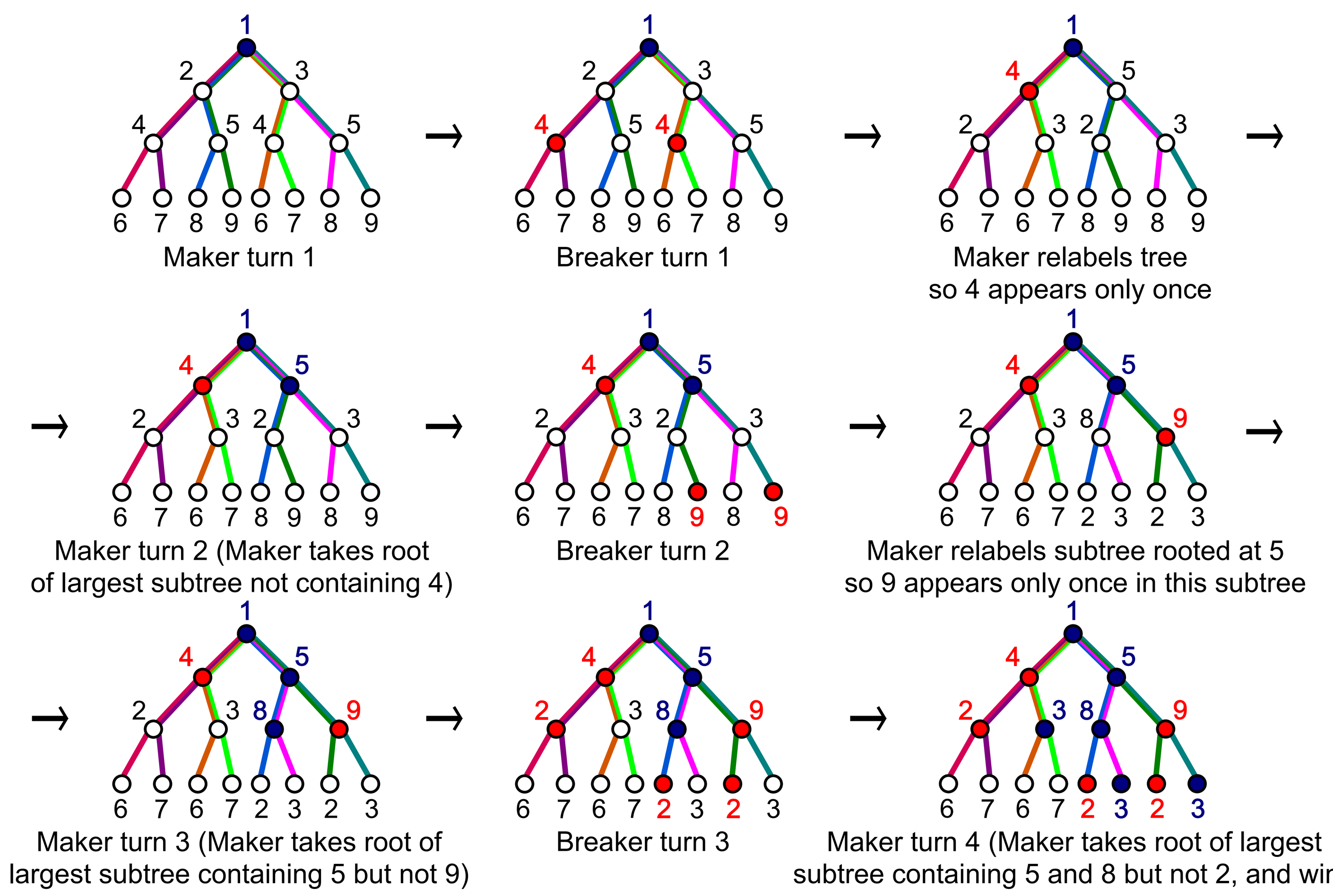
Theorem: If $\mathcal{H}$ is an n-uniform tumbleweed and L is a labeling of a complete n-level binary tree T which realizes $\mathcal{H}$, then L must be a good labeling.

A tumbleweed $\mathcal{H}$ has the **Single Label Property** (SLP) if for each vertex $v \in V(\mathcal{H})$, there exists a good labeling of a complete binary tree T which realizes $\mathcal{H}$ in which v appears as a label exactly once.



An example of a tumbleweed with the SLP. The labels 1, 2, and 3 appear once in the first labeling, and the labels 4 and 5 appear once in the second labeling.

Maker has an economical winning strategy on tumbleweeds with the SLP.



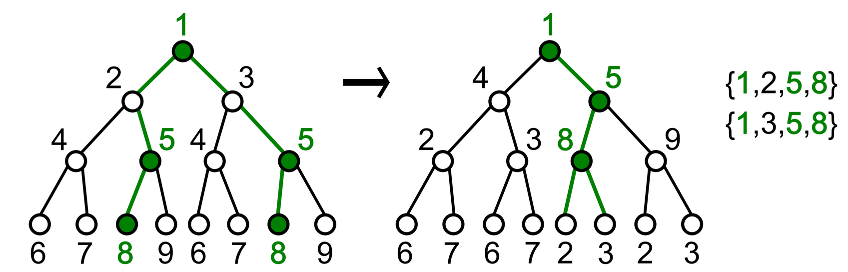
Theorem: An n-uniform hypergraph $\mathcal{H}$ is an economical extremal hypergraph for the Erdős–Selfridge theorem if and only if $\mathcal{H}$ is a tumbleweed with the SLP.

4. Second Characterization

For any hypergraph $\mathcal{H}$ and vertex $v \in V(\mathcal{H})$, let $\mathcal{H}(v) = \{A \in \mathcal{H} : v \in A\}$ be the set of edges in $\mathcal{H}$ which contain v.

Proposition: If $\mathcal{H}$ is an n-uniform tumbleweed with the Single Label Property, then $|\mathcal{H}(v)| = 2^{n-k}$ for each vertex $v \in V(\mathcal{H})$, where $k = \left|\bigcap_{A \in \mathcal{H}(v)} A\right|$.

Example:



A tumbleweed with the Single Label Property.

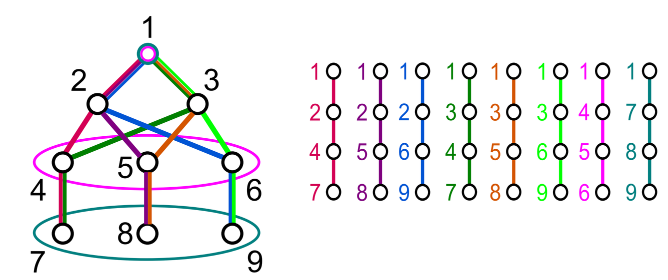
The label 8 appears twice in the first labeling, but the binary tree can be relabeled so that the label 8 appears exactly once. Since this label appears in level 3 of the tree, there are 3 vertices in common to every edge containing 8 (namely, 1, 5, and 8). So, since $|\mathcal{H}(8)| = 2^{4-3} = 2$, the desired property holds for the vertex 8. This process can be repeated with every vertex of the tumbleweed to prove the proposition.

Theorem: An n-uniform hypergraph $\mathcal{H}$ is an economical extremal hypergraph for the Erdős–Selfridge theorem if and only if $\mathcal{H}$ is a tumbleweed such that for all $v \in V(\mathcal{H})$, $|\mathcal{H}(v)| = 2^{n-k}$ where $k = \left|\bigcap_{A \in \mathcal{H}(v)} A\right|$.

5. Open Questions

Question 1: Characterize all extremal hypergraphs for the Erdős-Selfridge Theorem.

Some extremal hypergraphs cannot be realized by labelings of a complete binary tree:



An example given by Lu [3] of an extremal hypergraph which cannot not be realized by a labeling of a complete binary tree.

Question 2: Characterize all labelings of a complete binary tree which realize extremal hypergraphs for the Erdős-Selfridge Theorem.

6. Acknowledgements

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7. References

[1] József Beck. Inevitable randomness in discrete mathematics. *University Lecture Series, 49* American Mathematical Society, Providence, RI, xii+250, 2009.

[2] P. Erdős and J. L. Selfridge. On a combinatorial game. *J. Combinatorial Theory Ser.* A, 14:298-301, 1973.

[3] Xiaoyun Lu. A characterization on n-critical economical generalized tic-tac-toe games. *Discrete Math.* 110(1-3):197-203, 1992.