2022 Session B

B1. Suppose that $P(x) = a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial with integer coefficients, with a_1 odd. Suppose that $e^{P(x)} = b_0 + b_1x + b_2x^2 + \cdots$ for all x. Prove that b_k is nonzero for all $k \geq 0$.

Solution: Call a power series $\sum_{k=0}^{\infty} \frac{c_k}{k!} x^k$ even-ish if $c_0 = 1$ and all other c_k are even integers, and odd-ish if $c_0 = 1$ and all other c_k are odd integers. (If the series converges for x near 0, these definitions say that the associated function $f(x)$ has $f(0) = 1$ and for $k > 0$ all $f^{(k)}(0)$ even, respectively odd, integers.) Note that we have

$$\sum_{k=0}^{\infty} \frac{c_k}{k!} x^k \cdot \sum_{k=0}^{\infty} \frac{d_k}{k!} x^k = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{r=0}^{k} \binom{k}{r} c_r d_{k-r} x^k.$$

Therefore a product of even-ish series is again even-ish (for any $k \geq 1$ and any r at least one of c_r and d_{k-r} is even) and a product of an even-ish series and an odd-ish series is odd-ish (the only term contributing to the coefficient of x^k that is not even is $\binom{k}{0} c_0 d_k$, which is odd). Since

$$e^{a_1 x} = \sum_{k=0}^{\infty} \frac{a_1^k}{k!} x^k$$

is odd-ish, and for $j > 1$

$$e^{a_j x^j} = \sum_{k=0}^{\infty} \frac{a_j^k (jk)!/k!}{(jk)!} x^{jk}$$

is even-ish (since for $k \geq 1$, $(jk)!/k!$ is an even integer). Thus, by the remarks above, $e^{P(x)} = e^{a_1 x} \cdot e^{a_2 x^2} \cdot \cdots \cdot e^{a_n x^n}$ is odd-ish, and in particular its coefficients cannot be zero.

B2. Let × represent the cross product in $\mathbb{R}^3$. For what positive integers n does there exist a set $S \subset \mathbb{R}^3$ with exactly n elements such that

$$S = \{v \times w \colon v, w \in S\}?$$

Answer: $n = 1$ or $n = 7$.

Solution: Since we only care about positive n, we can assume S is nonempty. For any $v \in S$, we must have $v \times v = 0 \in S$. If $S = \{0\}$, then we have a solution with $n = 1$. Otherwise, let v_1 be a nonzero vector in S. Since v_1 must be in $\{v \times w \colon v, w \in S\}$, there must be vectors $v_2, v_3 \in S$ with $v_1 = v_2 \times v_3$. For $n \geq 3$, define $v_{n+1} = v_1 \times v_n \in S$. Since $v_1 \perp v_n$ for all $n \geq 3$, by induction on these n, we have $|v_n| = |v_1|^{n-3}|v_3|$. Since S is finite, this implies $|v_1| = 1$. Thus, all nonzero vectors in S have unit length.

Now choose a particular nonzero $v_1 \in S$, and as before, choose $v_2, v_3 \in S$ with $v_1 = v_2 \times v_3$. Then since $|v_2| = |v_3| = |v_2 \times v_3| = 1$, we know that v_2 and v_3 are orthogonal to each other as well as to v_1. Thus, $\{v_1, v_2, v_3\}$ forms an orthonormal basis of $\mathbb{R}^3$, and S contains all the cross products of two of them in either order, which means $\{0, \pm v_1, \pm v_2, \pm v_3\} \subset S$. Finally, if w is a nonzero vector in S, it has length 1, and its cross product with each v_i is either 0 or of length 1. It follows that $w = \pm v_i$ for some i. Hence the only possibilities are $n = 1$ and $n = 7$.

B3. Assign to each positive real number a color, either red or blue. Define a recoloring process as follows. First, let D be the set of all distances $d > 0$ such that there are two points of the same color at distance d apart. Second, recolor the positive reals so that the numbers in D are red and the numbers not in D are blue. If we iterate this recoloring process, will we always end up with all the numbers red after a finite number of steps?

Answer: Yes.

Solution 1: We first prove the following lemma.

Lemma. After a recoloring d and $2d$ cannot both be blue.

Proof. If d is blue after the recoloring, then in the original coloring any two points at distance d are opposite colors. But then for any x the sequence $x, x+d, x+2d, \ldots$ must alternate between red and blue. Hence there are pairs of points $2d$ apart of the same color and after the recoloring $2d$ will be red.

Now suppose that d is blue after two recolorings. Then after one recoloring the sequence $d, 2d, 3d, 4d$ had to alternate between red and blue. Since $2d$ and $4d$ cannot both be blue after one recoloring, they must both be red after one recoloring. Thus after one recoloring d and $3d$ must both be blue and $2d$, $4d$ must both be red.

Hence in the original coloring $d, 2d, 3d, 4d$ also alternate between red and blue. Thus the point $5d/2$ must have been the same color as one of $(2d, 3d)$ and as one of $(d, 4d)$. Thus both $d/2$ and $3d/2$ must be red after the first recoloring, and hence d must be red after the second recoloring. This is a contradiction. Thus all the numbers are red after two recolorings.

Solution 2:

We claim that all numbers are colored red after two recolorings.

Lemma. After a recoloring, if $d > 0$ is blue, then $d/2$ and $3d/2$ are both red.

Proof. By hypothesis, in the previous coloring, d and $2d$ must have had different colors. Also, $2d$ and $3d$ must have had different colors, and likewise for $3d$ and $4d$. Then d and $4d$ must have had different colors, so $5d/2$ must have had the same color as either d or $4d$, and since it is $3d/2$ away from each, $3d/2$ must be red after the recoloring. Similarly, $5d/2$ must have had the same color as either $2d$ or $3d$, and since it is $d/2$ away from each, $d/2$ must be red after the recoloring.

One consequence of the lemma is that d cannot be blue in two consecutive recolorings, since $d/2$ and $3d/2$ are distance d apart. Thus, if d is blue after two recolorings, is must have been red after one recoloring. Then by the reasoning in the proof of the lemma, $2d$ must have been blue after one recoloring, so $3d$ must have been red, so $4d$ must have been blue. But by the lemma, since $4d$ was blue after one recoloring, $2d = 4d/2$ must have been red, a contradiction. Therefore, no d can be blue after two recolorings.

B4. Find all integers n with $n \geq 4$ for which there exists a sequence of distinct real numbers $x_1, \ldots, x_n$ such that each of the sets

$$\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \ldots, \{x_{n-2}, x_{n-1}, x_n\}, \{x_{n-1}, x_n, x_1\}, \text{ and } \{x_n, x_1, x_2\}$$

forms a 3-term arithmetic progression when arranged in increasing order.

Answer: $n = 9, 12, 15, 18, \ldots$; more precisely, all multiples of 3 that are strictly greater than 6.

Solution: Since $\{x_j, x_{j+1}, x_{j+2}\}$ forms an arithmetic progression after reordering, we must have $x_{j+1} + x_{j+2} = 2x_j$, or $x_j + x_{j+2} = 2x_{j+1}$, or $x_j + x_{j+1} = 2x_{j+2}$. Thus, x_{j+2} must be equal to one of $2x_j - x_{j+1}$, $2x_{j+1} - x_j$, or $(x_j + x_{j+1})/2$. Hence $x_{j+2} - x_{j+1}$ must be equal to one of $-2(x_{j+1} - x_j)$, $x_{j+1} - x_j$, or $-(x_{j+1} - x_j)/2$. Thus, by an easy induction, there will be a sequence of integers k_j such that $x_{j+1} - x_j = (-2)^{k_j}(x_2 - x_1)$ for $1 \leq j \leq n-1$.

The statements in the previous paragraph are also true for the triples $\{x_{n-1}, x_n, x_1\}$ and $\{x_n, x_1, x_2\}$, and in particular, $x_1 - x_n = (-2)^{k_n}(x_2 - x_1)$ for some integer k_n. Notice also that $|k_{j+1} - k_j| \leq 1$ for $1 \leq j \leq n-1$, that $|k_1 - k_n| \leq 1$, and that $k_1 = 0$. Cyclically rotating the sequence, and rescaling (possibly by a negative number), we may assume that $x_2 - x_1 = 1$ and that all the k_j are nonnegative. Then

$$0 = (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) + (x_1 - x_n) = \sum_{j=1}^{n} (-2)^{k_j} \equiv \sum_{j=1}^{n} 1 = n \pmod 3.$$

Hence n must be a multiple of 3.

For $m \geq 2$ and $n = 3m + 3$, an example of such a sequence is

$$(1, 3, 5, \ldots, 4m-3, 4m-1, 4m-2, 4m, 4m-4, \ldots, 8, 4, 0, 2).$$

Notice that this sequence consists, aside from $4m - 2$ and 2, of an increasing subsequence of odd numbers and a decreasing subsequence of multiples of 4. Thus, the elements of the sequence are distinct unless $4m - 2 = 2$, which happens only for $m = 1$. In fact, there is no example for $n = 6$. (If, as above, we arrange that $x_2 - x_1 = 1$ and all the differences $x_{j+1} - x_j$ and $x_1 - x_n$ are in $\{1, -2, 4, \ldots\}$, then by parity we must have an even number equal to 1. We cannot have three consecutive differences be $1, 1, -2$ or $1, -2, 1$, since either would give two equal terms three apart. Since a 1 can only be adjacent to another 1 or a -2, it follows that we cannot have differences of 1 that are adjacent or two apart. Then we must have exactly two differences of 1, three apart, and the only remaining possible difference sequence is $1, -2, -2, 1, -2, -2$, which does not sum to 0.) Therefore, the answer is that n can be any multiple of 3 that is strictly greater than 6.

B5. For $0 \le p \le 1/2$, let $X_1, X_2, \dots$ be independent random variables such that

$$X_i = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } p, \\ 0 & \text{with probability } 1-2p, \end{cases}$$

for all $i \ge 1$. Given a positive integer n and integers $b, a_1, \dots, a_n$, let $P(b, a_1, \dots, a_n)$ denote the probability that $a_1X_1 + \cdots + a_nX_n = b$. For which values of p is it the case that $P(0, a_1, \dots, a_n) \ge P(b, a_1, \dots, a_n)$ for all positive integers n and all integers $b, a_1, \dots, a_n$?

Answer: $0 \le p \le 1/4$.

Solution: For each $0 \le p \le 1/4$, there is a q between 0 and $1/2$ (inclusive) for which $q(1-q) = p$. For $1 \le k \le n$, let Y_k and Z_k be independent random variables such that $Y_k = 1/2$ with probability q and $Y_k = -1/2$ with probability $1-q$, and Z_k has the same distribution as Y_k. Then X_k can be expressed as $Y_k - Z_k$. Let $Y = a_1Y_1 + \cdots + a_nY_n$, and let $Z = a_1Z_1 + \cdots a_nZ_n$; then $a_1X_1 + \cdots + a_nX_n = Y - Z$, and Y and Z have the same distribution. This distribution has probabilities $p_{-m}, p_{-m+1}, \dots, p_m$ where $m = (|a_1| + \cdots + |a_n|)/2$; notice that the indices are half-integers if m is a half-integer. Define $p_j = 0$ for every index j with $|j| > m$. Then for every integer b, the Cauchy-Schwarz inequality implies that

$$P(b, a_1, \dots, a_n) = \sum_{j=-m}^{m} p_{j+b}p_j \le \sum_{j=-m}^{m} p_j^2 = P(0, a_1, \dots, a_n).$$

To show that the inequality need not be true for $p > 1/4$, let $a_k = 2^{k-1}$ for $k \ge 1$. Then $a_1X_1 + \cdots + a_nX_n = 0$ only when $X_1 = \cdots = X_n = 0$, which occurs with probability $P(0, a_1, a_2, \dots, a_n) = (1-2p)^n$. For a particular choice of $X_1, \dots, X_n$ that are not all zero, let m be the largest index for which $X_m \ne 0$. Then if $a_1X_1 + \cdots + a_nX_n = 1$, we must have $X_m = 1$ and $X_k = -1$ for $1 \le k < m$. The probability of this event (including the fact that $X_{m+1} = \cdots = X_n = 0$) is $p^m(1-2p)^{n-m}$. Thus, $P(1, a_1, a_2, \dots, a_n) = p(1-2p)^{n-1} + p^2(1-2p)^{n-2} + \cdots + p^n$.

If $p > 1/4$, we claim that $P(1, a_1, a_2, \dots, a_n) > P(0, a_1, a_2, \dots, a_n)$ for some n. Indeed this is true for $n = 1$ if $p > 1/3$ and for $n = 2$ if $p = 1/3$, so henceforth we assume $1/4 < p < 1/3$. Then

$$\begin{aligned} P(1, a_1, a_2, \dots, a_n) &= p(1-2p)^{n-1} \sum_{k=0}^{n-1} [p/(1-2p)]^k \\ &= p(1-2p)^{n-1} \frac{1 - [p/(1-2p)]^n}{1 - p/(1-2p)} \\ &= (1-2p)^n \frac{p}{1-3p} (1 - [p/(1-2p)]^n). \end{aligned}$$

Since $p > 1/4$, we have $p/(1-3p) > 1$, and since $p < 1/3$, we have $p/(1-2p) < 1$. Thus, for n sufficiently large, $1 - [p/(1-2p)]^n > (1-3p)/p$, and hence $P(1, a_1, a_2, \dots, a_n) > (1-2p)^n = P(0, a_1, a_2, \dots, a_n)$ as claimed.

B6. Find all continuous functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(xf(y)) + f(yf(x)) = 1 + f(x+y)$$

for all $x, y > 0$.

Answer: $f(x) = 1/(1+cx)$ where c is a nonnegative real number.

Solution 1: Suppose we have $f(a) = f(a') = 1$, then plugging $(x,y) = (a,a')$ into the functional equation, we find $f(a+a') = 1$. In particular, if there is any a with $f(a) = 1$, we find that $f(na) = 1$ for all $a \in \mathbb{N}$.

Suppose there is some closed interval $[a,a']$ with $f(x) = 1$ for all $a \le x \le a'$. Using the remark above we see that $f(x) = 1$ for all $na \le x \le na'$ and any positive integer n. Choosing n large enough that $n(a'-a) > a$, we find that successive intervals $[na, na']$ and $[(n+1)a, (n+1)a']$ overlap. Hence there is some $b > 0$ such that $f(x) = 1$ for all $x > b$. For any x choose y large enough that $yf(x) > b$ and $x + y > b$. Then the functional equation gives $f(x) = 1$. Thus f is identically equal to 1.

Now suppose that there is some $c > 0$ with $f(c) > 1$. Define $u = \frac{c}{f(c)-1}$, so that $uf(c) = u + c$. Then the functional equation for $(x,y) = (u,c)$ gives $f\left(cf\left(\frac{c}{f(c)-1}\right)\right) = 1$. In particular, setting $a = cf\left(\frac{c}{f(c)-1}\right)$ we have $f(a) = 1$ and hence $f(na) = 1$ for all n. Since we can choose $na > c$, there is a least $d > c$ such that $f(d) = 1$. On the interval $[c,d)$ the expression $\frac{x}{f(x)-1}$ goes from u (at $x = c$) to infinity (as $x \to d^-$). Thus we can choose $x_m \in [c,d)$ with $\frac{x_m}{f(x_m)-1} = ma$ for all m with $ma > u$. Hence at $x = x_m$, $xf\left(\frac{x}{f(x)-1}\right)$ takes on the value x_m. But this says that $xf\left(\frac{x}{f(x)-1}\right)$ is a non-constant continuous function on $[c,d)$, hence it takes on every value in some open interval I. Since $f > 1$ on $[c,d)$, using the argument at the beginning of this paragraph with c replaced by x shows that $f(a) = 1$ for all $a \in I$. Hence by the previous paragraph, we have f identically 1, and a contradiction. Thus $f(x) \le 1$ for all x.

Suppose $f(a) = 1$ for some a. Then for any $x \in (0, na)$ (with $n \ge 3$) we have $f(xf(na - x)) + f((na-x)f(x)) = 1 + f(na) = 2$ and hence $f(xf(na-x)) = 1$ for all x. However, taking $x = a, 2a$ we find that $xf(na-x)$ takes on the values a and $2a$ and hence again this gives an interval on which f equals 1. Hence if $f(a) = 1$ for any a, then f is identically equal to 1.

Thus we may suppose $f(x) < 1$ for all $x \in (0,\infty)$. Note that this implies $f(x+y) < f(yf(x))$ for all $x, y > 0$. For any $x_0 \in (0,\infty)$ define a sequence by $x_{n+1} = \frac{x_n}{2} f\left(\frac{x_n}{2}\right)$ for $n \ge 1$. Then $x_{n+1} < \frac{x_n}{2}$, so this sequence decreases to 0, and the functional equation gives $f(x_{n+1}) = \frac{1+f(x_n)}{2}$, so $f(x_n)$ converges to 1.

Now we will show that f is strictly decreasing. Suppose $c < d$. The expression $(d-x)f(x)$ tends to d if we take $x = x_n$ and $n \to \infty$ and tends to 0 as $x \to d$. Thus there is some $x \in (0,d)$ with $(d-x)f(x) = c$. Taking $y = d - x$ in the functional equation, we get $x + y = d$ and $yf(x) = c$, and hence $f(c) > f(d)$. Thus f is strictly decreasing. Note that this argument shows that for any $c < d$, we can choose x, y such that $x + y = d$ and $yf(x) = c$, and hence we will get $f(xf(y)) + f(c) = 1 + f(d)$.

Now fix any x_0 and look at the sequence defined by $x_{n+1} = \frac{x_n}{2} f\left(\frac{x_n}{2}\right)$. Then we have $f(x_{n+1}) = \frac{1+f(x_n)}{2}$ and hence $f(x_n) = 1 - 2^{-n}(1 - f(x_0))$. Since f is decreasing it follows that $\lim_{x\to 0^+} f(x) = 1$, and hence we can (and will) extend f continuously to $[0,\infty)$. Further since $f < 1$ is decreasing, we have

$$\frac{x_n}{2} > x_{n+1} > \frac{x_n}{2} f(x_n) = \frac{x_n}{2}\left(1 - 2^{-n}(1 - f(x_0))\right),$$

and hence

$$2^{-n}x_0 > x_n > 2^{-n}x_0 \prod_{k=0}^{n-1}\Big(1 - 2^{-k}(1 - f(x_0))\Big) > 2^{-n}x_0 \prod_{k=0}^{\infty}\Big(1 - 2^{-k}(1 - f(x_0))\Big).$$

Since the product converges, we get a positive constant C with $2^{-n}x_0 > x_n > C2^{-n}x_0$. Since the graph of $y = f(x)$ lies in the union of the rectangles $[x_{n+1}, x_n] \times [f(x_n), f(x_{n+1})]$, we have that for any $x \in [x_{n+1}, x_n]$

$$\frac{1 - f(x)}{x} \le \frac{1 - f(x_n)}{x_{n+1}} \le \frac{2^{-n}(1 - f(x_0))}{2^{-n-1}Cx_0} = \frac{2(1 - f(x_0))}{Cx_0}.$$

Hence the (negated) slopes of secant lines to $y = f(x)$ through $(0,1)$ are bounded and we can write

$$0 \le \alpha = \liminf_{x\to 0^+} \frac{1 - f(x)}{x} \le \limsup_{x\to 0^+} \frac{1 - f(x)}{x} = \beta < \infty.$$

Suppose the derivative at 0 does not exist, so that $\alpha < \beta$. Choose an x_0 small enough that $f(x_0) > 2/3$ and $\frac{1-f(x_0)}{x_0} = \beta - \epsilon$ for some small ϵ, and consider the sequence above. Then we have $x_{n+1} > \frac{x_n}{2} f(x_0) > \frac{x_n}{3}$ and since the slopes $\frac{1-f(x_n)}{x_n}$ are increasing, we have $\frac{1-f(x_n)}{x_n} \ge \beta - \epsilon$. Now for any sequence c_n tending to 0 with $\frac{1-f(c_n)}{c_n} < \gamma = \frac{\beta+\alpha}{2}$, we can choose d_n to be the next larger term in the sequence (x_k) and have $d_n/3 < c_n < d_n$. By the above we can choose x'_n, y'_n (which will tend to 0 since $x'_n, y'_n < d_n < 3c_n$) with $x'_n + y'_n = d_n$ and $y'_n f(x'_n) = c_n$ and hence, we get

$$\begin{aligned}\frac{1 - f(x'_n f(y'_n))}{x'_n f(y'_n)} &= \frac{(1 - f(d_n)) - (1 - f(c_n))}{\left(d_n - \frac{c_n}{f(x'_n)}\right) f(y'_n)} \ge \frac{(\beta - \epsilon)d_n - \gamma c_n}{\left(d_n - \frac{c_n}{f(x'_n)}\right) f(y'_n)} \\ &= \frac{\beta - \epsilon}{f(y'_n)} + \frac{\left(\frac{\beta-\epsilon}{f(x'_n)} - \gamma\right) c_n}{\left(d_n - \frac{c_n}{f(x'_n)}\right) f(y'_n)} \ge \frac{\beta - \epsilon}{f(y'_n)} + \frac{\frac{\beta-\epsilon}{f(x'_n)} - \gamma}{\left(3 - \frac{1}{f(x'_n)}\right) f(y'_n)}.\end{aligned}$$

As n tends to infinity, this lower bound tends to $\beta - \epsilon + \frac{\beta-\gamma-\epsilon}{2}$. But for sufficiently small ϵ this exceeds β, contradicting the definition of β. Thus $f'(0) = -\beta$ exists.

Now for any $c < d$ with $d - c$ small, we again choose x, y with $x + y = d$ and $yf(x) = c$. Hence $x = d - y < d - yf(x) = d - c$ is small. Then

$$\frac{f(c) - f(d)}{d - c} = \frac{1 - f(xf(y))}{d - c} = \frac{1 - f(xf(y))}{xf(y)} \cdot \frac{f(y)}{1 + y\frac{1-f(x)}{x}}.$$

Taking the limit as c, d both converge to some fixed t, we see that y also converges to t and x converges to 0. We conclude that f is differentiable at t and

$$f'(t) = -\frac{\beta f(t)}{1 + \beta t}.$$

Thus f is continuously differentiable and satisfies the differential equation above with $f(0) = 1$. Thus $f(x) = \frac{1}{1+\beta x}$ for some $\beta \geq 0$ and these are indeed solutions.

Solution 2: The only solutions are $f(x) = 1/(1+cx)$ for all $x \in \mathbb{R}^+$, where c is a nonnegative real number. Checking that these functions are solutions is straightforward. To prove that these are the only solutions, we start with some notation and a technical lemma. Let D_- and D_+ denote the derivatives of a function from the left and right, respectively.

Lemma 1. *If* $g : \mathbb{R}^+ \to \mathbb{R}$ *is a continuous function for which* $D_-g(z) = \lim_{w \to z^-}[g(z) - g(w)]/(z - w)$ *exists and equals* 0 *for all* $z \in \mathbb{R}^+$*, then* g *is a constant function.*

Proof. If g is not constant, choose $a, b \in \mathbb{R}^+$ such that $g(a) \neq g(b)$, and without loss of generality assume that $a < b$ and $g(a) < g(b)$. Let $s = [g(b) - g(a)]/(b - a)$. The continuous function $g(x) - sx$ has a maximum value on $[a, b]$, and since $g(a) - sa = g(b) - sb$, this maximum value is attained at at least one point in $[a, b]$ other than a; thus, there exists $z \in (a, b]$ such that $g(w) - sw \leq g(z) - sz$ for all $w \in [a, b]$. Then $[g(z) - g(w)]/(z - w) \geq s > 0$ for all $w \in [a, z)$, contradicting the hypothesis. $\square$

The next lemma moves the goalposts closer. This and all lemmas below assume the hypotheses of the problem in addition to the hypotheses stated in the lemma.

Lemma 2. *If both of the following limits exist,* $\lim_{x \to 0^+} f(x) = 1$*, and* $\lim_{x \to 0^+}[1 - f(x)]/x = c \geq 0$*, then* $f(x) = 1/(1 + cx)$ *for all* $x \in \mathbb{R}^+$*.*

Proof. Choose $x, z \in \mathbb{R}^+$ with $x < z$, and let $y = z - x$ and $w = yf(x)$. Then

$$\lim_{x \to 0^+} \frac{z - w}{x} = \lim_{x \to 0^+} \frac{z - (z - x)f(x)}{x} = \lim_{x \to 0^+} \frac{z[1 - f(x)] + xf(x)}{x} = cz + 1.$$

In particular, $z - w$ is positive and a one-to-one function of x for $x > 0$ sufficiently small, so taking a limit as $x \to 0+$ is equivalent to taking a limit as $w \to z^-$.

Next, from the identity in the problem statement, $f(z) - f(w) = f(x + y) - f(yf(x)) = f(xf(y)) - 1 = f(xf(z - x)) - 1$. Then, using the fact that $f(z - x)$ is a continuous function of (sufficiently small) x,

$$D_-f(z) = \lim_{w \to z^-} \frac{f(z) - f(w)}{z - w} = \lim_{x \to 0+} \frac{f(xf(z - x)) - 1}{xf(z - x)} \cdot \frac{xf(z - x)}{z - w} = -c\frac{f(z)}{1 + cz},$$

Let $g(x) = \log f(x) + \log(1 + cx)$ for all $x \in \mathbb{R}^+$. The chain rule applies to derivatives from the left when the outer function is two-sided differentiable, so $D_-g(z)$ exists for all $z \in \mathbb{R}^+$, and $D_-g(z) = -c/(1 + cz) + c/(1 + cz) = 0$. By Lemma 1, g is constant, and since $\lim_{x \to 0+} g(x) = 0$, we have $g(x) = 0$ for all $x \in \mathbb{R}^+$. If follow that $f(x) = \exp[g(x) - \log(1 + cx)] = 1/(1 + cx)$ as claimed. $\square$

The remainder of the proof establishes the two limits in the hypothesis of Lemma 2.

Lemma 3. *Given the hypotheses of the problem,* $\limsup_{x \to 0^+} f(x) \geq 1$.

Proof. If $\limsup_{x \to 0^+} f(x) < 1$, then for some $x_0 > 0$ and some $r < 1$, we have $f(x) \leq r$ for $0 < x \leq x_0$. Let $x_{n+1} = x_n f(x_n/2)/2$ for $n \geq 0$. Then, by induction, $x_0 > x_1 > \cdots$ and $f(x_n) \leq r$ for all $n \geq 0$. But with $x = y = x_n/2$, the identity in the problem statement yields $2f(x_{n+1}) = 1 + f(x_n)$, which implies that $f(x_n) \to 1$ as $n \to \infty$, a contradiction. $\square$

Lemma 4. *For all $a, b \in \mathbb{R}^+$ with $a < b$, there exists $z(a,b) \in \mathbb{R}$ such that $f(z(a,b)) = 1 + f(b) - f(a)$.*

Proof. By Lemma 3, $\limsup_{x \to 0^+} (b-x)f(x) \geq b$, and $(b-x)f(x)$ is a continuous function of f with value 0 at $x = b$. Since $0 < a < b$, it follows from the intermediate value theorem that $(b-x)f(x) = a$ for some $x \in (0, b)$. Let $y = b - x$ and apply the identity in the problem statement to get $f(xf(b-x)) + f(a) = 1 + f(b)$. Thus, the lemma is proved with $z(a,b) = xf(b-x)$. □

Lemma 5. *If $f(z) = 1$ for some $z \in \mathbb{R}^+$, then $f(x) = 1$ for all $x \in (0, z)$.*

Proof. If $f(x) \neq 1$ for some $x \in (0, z)$, then let $z_0 = z$ and $z_1 = x$, and define z_{n+1} for $n \geq 1$ in terms of $z_0, z_1, \ldots, z_n$ as follows. Let $M_n = \max_{0 \leq n} f(z_k)$ and $m_n = \min_{0 \leq n} f(z_k)$. Choose integers i and j with $0 \leq i, j < n$ such that $f(z_i) = m_n$ and $f(z_j) = M_n$, and let $a_n = \min(z_i, z_j)$ and $b_n = \max(z_i, z_j)$. Then $|f(b_n) - f(a_n)| = M_n - m_n$. Notice that $f(z_0) \neq f(z_1)$ implies that $f(z_i) < f(z_j)$, which implies that $z_i \neq z_j$, which implies that $a_n < b_n$. Let $z_{n+1} = z(a_n, b_n)$.

We will prove by induction that $M_n - m_n \geq n|f(x) - 1|$. Notice that $M_0 - m_0 = 0$ and that $M_1 - m_1 = |f(z_1) - f(z_0)| = |f(x) - f(z)| = |f(x) - 1|$. Next, since $z_1 = x < z = z_0$, we have $a_1 = x$ and $b_1 = z$, so by Lemma 4, $f(z_2) = f(z(a_1, b_1)) = 1 + f(z) - f(x) = 2 - f(x)$. Thus, $|f(z_2) - 1| = |f(x) - 1|$; also, $f(z_2)$ and $f(z_1) = f(x)$ lie on opposite sides of 1. Then $M_2 - 1 = 1 - m_2 = |f(x) - 1|$, from which it follows immediately that $M_n - 1 \geq |f(x) - 1|$ and $1 - m_n \geq |f(x) - 1|$ for $n \geq 2$. Also, $M_2 - m_2 = 2|f(x) - 1|$.

Assume now for some $n \geq 2$ that $M_n - m_n \geq n|f(x) - 1|$. Recall that $f(b_n)$ is either M_n or m_n. If $f(b_n) = M_n$ then by Lemma 4, $f(z_{n+1}) - f(b_n) = f(z(a_n, b_n)) - f(b_n) = 1 - f(a_n) = 1 - m_n$. Then $m_{n+1} = m_n$ and $M_{n+1} = f(z_{n+1}) = M_n + 1 - m_n \geq M_n + |f(x) - 1|$, so $M_{n+1} - m_{n+1} \geq M_n - m_n + |f(x) - 1| \geq (n+1)|f(x) - 1|$, as desired. If $f(b_n) = m_n$, a similar argument completes the induction.

We have now proved that $M_n - m_n$ becomes arbitrarily large as n increases, and since $m_n > 0$, in fact M_n becomes arbitrarily large. In particular, $M_n \geq 2$ for some n; then since M_n is a value of f and $f(z) = 1$, by the intermediate value theorem there exists $w \in \mathbb{R}^+$ such that $f(w) = 2$. Then letting $x = y = w$ in the identity in the problem statement, we have $2f(2w) = 1 + f(2w)$, so $f(2w) = 1$. Then applying Lemma 4, we have that $f(z(w, 2w)) = 1 + 1 - 2 = 0$. But this contradicts the hypotheses of the problem. □

Next, suppose that $f(x) > 1$ for some $x \in \mathbb{R}^+$. Let $y = x/(f(x) - 1)$, so that $x + y = yf(x)$. Then by the identity in the problem statement, $f(xf(y)) = 1$, so Lemma 5 applies. Furthermore, if Lemma 5 applies, then the hypotheses of Lemma 2 apply with $c = 0$. Since it suffices to verify the hypotheses of Lemma 2, we can assume for the rest of the proof that $f(x) < 1$ for all $x \in \mathbb{R}^+$.

Lemma 6. *In the case that $f(x) < 1$ for all $x \in \mathbb{R}^+$, we have $\lim_{x \to 0^+} f(x) = 1$, and f is strictly decreasing on $\mathbb{R}^+$.*

Proof. The first statement follows immediately from the hypothesis and Lemma 3. Next, if $f(a) = f(b)$ for some $a < b$, then by Lemma 4, $f(z(a,b)) = 1$, contradicting the hypothesis. Thus, f is strictly monotonic, and because of the limit at 0, it cannot be increasing, so it is decreasing. □

Lemma 6 verifies the first limit in the hypotheses for Lemma 2. Let $f(0) = 1$, so that f is continuous for $x \geq 0$. It remains to show that $D_+ f(0)$ exists; then it must be nonpositive since f is decreasing, making c in Lemma 2 nonnegative.

Lemma 6 and $f(0) = 1$ imply that $h(u) = f^{-1}(1-u)$ is well defined and strictly increasing for $u \geq 0$ sufficiently small, with $h(0) = 0$. It suffices to show that $D_+ h(0)$ exists and is positive, whereupon $D_+ f(0) = -1/D_+ h(0)$.

Lemma 7. *For $v \geq u \geq 0$ with v in the domain of h,*

$$(1-v)h(v) \leq h(u) + h(v-u) \leq h(v).$$

Proof. The inequalities follow directly from $h(0) = 0$ in the cases $u = 0$ and $u = v$, so assume henceforth that $0 < u < v$. Let $b = h(v)$. As in the proof of Lemma 4, consider $(b-x)f(x)$ as a function of $x \in [0, b]$. By Lemma 6, this is a strictly decreasing continuous function with values b at $x = 0$ and 0 at $x = b$. Then $f((b-x)f(x))$ is strictly increasing and continuous in x, with values $f(b) = 1-v$ at $x = 0$ and $f(0) = 1$ at $x = b$. Thus, there is a (unique) $x \in (0, b)$ such that $f((b-x)f(x)) = 1-u$. Then $h(u) = (b-x)f(x)$. Applying the identity in the problem statement with $y = b-x$, we have $f(xf(b-x)) = 1 + f(b) - f((b-x)f(x)) = 1-v+u$, so $h(v-u) = xf(b-x)$. Then $h(u) + h(v-u) = (b-x)f(x) + xf(b-x) < b-x+x = b = h(v)$, as desired, and since $f(x)$ and $f(b-x)$ exceed $f(b) = 1-v$, we have $h(u) + h(v-u) > (b-x+x)(1-v) = (1-v)h(v)$, finishing the proof. $\square$

We claim that for all $w > 0$ in the domain of h and all $u \in (0, w]$,

$$(1-13w)h(w)/w = h(w)/w - 13h(w) < h(u)/u < h(w)/w + 13h(w) = (1+13w)h(w)/w.$$

This implies that $\ell = \liminf_{u \to 0^+} h(u)/u \geq (1-13w)h(w)/w$ and $L = \limsup_{u \to 0^+} h(u)/u \leq (1+13w)h(w)/w$. For $w < 1/13$, the lower bound on ℓ is positive, and $L/\ell \leq (1+13w)/(1-13w)$. Letting $w \to 0$, we conclude that $0 < \ell = L = \lim_{u \to 0^+} h(u)/u = D_+ h(0)$, as desired. It remains only to prove our claim.

For $k \geq 0$, applying Lemma 7 with $v = w/2^k$ and $u = w/2^{k+1} = v/2$, we have $(1 - w/2^k)h(w/2^k) \leq 2h(w/2^{k+1}) \leq h(w/2^k)$. By induction, $[h(w)/2^k] \prod_{j=0}^{k-1}(1 - w/2^j) \leq h(w/2^k) \leq h(w)/2^k$ for $k \geq 0$. Using concavity of the logarithm,

$$\begin{aligned}\prod_{j=0}^{k-1}(1 - w/2^j) &= \exp\left(\sum_{j=0}^{k-1} \log(1 - w/2^j)\right) \geq \exp\left(\sum_{j=0}^{k-1} \log(1-w)/2^j\right) \\ &> \exp\left(2\log(1-w)\right) > \exp\left(\log(1-2w)\right) = 1-2w.\end{aligned}$$

Thus,

$$(1-2w)h(w)/2^k \leq h(w/2^k) \leq h(w)/2^k.$$

If our claim is false, we can choose $u_0 \in (0, w]$ such that $|h(u_0)/u_0 - h(w)/w| \geq 13h(w)$. We will construct a sequence $\{u_j\}$ and prove by induction that for $j \geq 0$,

$$|h(u_j) - u_j h(w)/w| \geq (1 + 3 \cdot 2^{2-j})h(w)u_0.$$

For $j = 0$, this inequality is equivalent to the assumption for u_0. Notice that the right side is bounded below by $h(w)u_0$ for all j, while the left side approaches 0 as $u_j \to 0$. We will obtain a contradiction by showing as part of the induction that u_j becomes arbitrarily small.

To proceed inductively, assume for some $n \geq 0$ that real numbers $u_0 > u_1 > \cdots > u_n > 0$ and integers $0 \leq k_0 < k_1 < \ldots < k_n$ have been chosen so that for $j = 0, 1, \ldots, n$, the inequality displayed above holds and $w/2^{k_j} \geq u_j > w/2^{k_j+1}$. We can choose such a k_j for each $u_j \in (0, w]$, and choosing k_0 finishes the base case; the monotonicity of the sequences will be established as part of the induction. Let $u_{n+1} = w/2^{k_n} - u_n$. Then $0 \leq u_{n+1} < w/2^{k_n} - w/2^{k_n+1} = w/2^{k_n+1} < u_n$. Applying Lemma 7 with $u = u_n$ and $v = w/2^{k_n}$ yields $(1 - w/2^{k_n})h(w/2^{k_n}) \leq h(u_n) + h(u_{n+1}) \leq h(w/2^{k_n})$. Then using the inequalities displayed above,

$$\begin{aligned} |h(u_{n+1}) - u_{n+1}h(w)/w| &= |h(u_{n+1}) - (w/2^{k_n} - u_n)h(w)/w| \\ &= |h(u_{n+1}) + h(u_n) - h(w)/2^{k_n} - [h(u_n) - u_n h(w)/w]| \\ &\geq |h(u_n) - u_n h(w)/w| - |h(u_{n+1}) + h(u_n) - h(w)/2^{k_n}| \\ &\geq (1 + 3 \cdot 2^{2-n})h(w)u_0 - |h(u_{n+1}) + h(u_n) - h(w/2^{k_n})| \\ &\qquad - |h(w/2^{k_n}) - h(w)/2^{k_n}| \\ &\geq (1 + 3 \cdot 2^{2-n})h(w)u_0 - wh(w/2^{k_n})/2^{k_n} - 2wh(w)/2^{k_n} \\ &\geq (1 + 3 \cdot 2^{2-n})h(w)u_0 - 3wh(w)/2^{k_n}. \end{aligned}$$

Notice that $k_n \geq k_0 + n$ and $w/2^{k_0} < 2u_0$, so that $w/2^{k_n} < 2^{1-n}u_0$. Thus, $|h(u_{n+1}) - u_{n+1}h(w)/w| > (1 + 3 \cdot 2^{2-n} - 3 \cdot 2^{1-n})h(w)u_0 = (1 + 3 \cdot 2^{2-(n+1)})h(w)u_0$, as desired. Next, we observe that $u_{n+1} = 0$ would contradict the inequality we just proved, so $u_{n+1} > 0$, whence k_{n+1} is well-defined. Finally, we have already shown that $u_{n+1} < w/2^{k_n+1} < u_n$, so $k_{n+1} \geq k_n + 1$, and monotonicity of both sequences continues. With the induction complete, observe that $u_j \leq w/2^{k_j} \leq 2^{1-j}u_0$, which approaches 0 as $j \to \infty$, which contradicts the inequality the induction just established. Thus, our claim is true, and our solution is complete.

Solution 3: For positive numbers x and z, let $y = z/f(x)$. Then

$$f\left(xf\left(\frac{z}{f(x)}\right)\right) + f(z) = 1 + f\left(x + \frac{z}{f(x)}\right). \tag{*}$$

Since f is continuous, the right side approaches $1 + f(x)$ as $z \to 0^+$, and in particular is bounded for (positive) z in a neighborhood of 0. Then since f is positive-valued, both terms on the left side are bounded, and in particular $f(z)$ is bounded, for z near 0.

Let $g(z) = zf(z/2)/2$ for $z > 0$; boundedness of f near 0 implies that $g(z) \to 0^+$ as $z \to 0^+$. Then since g is continuous, $f(g(z))$ and $f(z)$ have the same $\liminf$ L_- and the same $\limsup$ L_+ as $z \to 0^+$, both of which are finite according to the preceding paragraph. Letting $x = y = z/2$ in the original functional equation and dividing by 2 yields

$$f(g(z)) = \frac{1 + f(z)}{2}.$$

Taking the $\liminf$ and $\limsup$ of this equation as $z \to 0^+$, it follows that $L_- = L_+ = 1$, so $f(z) \to 1$ as $z \to 0^+$. Define $f(0) = 1$, so that f is now continuous at 0.

Claim. *Further, f is differentiable from the right at 0.*

Proof. Choose z_0 sufficiently small that $|f(z) - 1| \leq 1/3$ for $z \in (0, z_0]$. For such z, we have $2/3 \leq f(z/2) \leq 4/3$, and hence

$$z/3 \leq g(z) \leq 2z/3.$$

Let $z_n = g(z_{n-1})$ for $n \geq 1$; then $z_0 > z_1 > z_2 > \cdots$, and $z_n \to 0^+$ as $n \to \infty$. By continuity, $g(z)$ takes on every value between z_{n+1} and z_n as z goes from z_n to z_{n-1}. Let w_n be a number in $[z_{n+1}, z_n]$; by induction on n, there is an associated sequence (not necessarily unique) of numbers $w_k \in [z_{k+1}, z_k]$ with $w_{k+1} = g(w_k)$ for $0 \leq k < n$. Since $|f(g(z)) - 1| = |f(z) - 1|/2$, it follows also by induction that $|f(w_n) - 1| \leq 2^{-n}/3$ for all $w_n \in [z_{n+1}, z_n]$. Here and for the rest of the proof, we regard z_n as fixed for all $n \geq 0$, but we regard w_n as an arbitrary number in $[z_{n+1}, z_n]$, with a corresponding sequence of predecessors w_k as described above.

Next, let $h(z) = (f(z) - 1)/z$ for $z > 0$. Then

$$h(g(z)) = \frac{f(g(z)) - 1}{g(z)} = \frac{(f(z) - 1)/2}{zf(z/2)/2} = \frac{h(z)}{f(z/2)}.$$

So for $n \geq m \geq 0$,

$$h(w_m) = h(w_n) \prod_{k=m}^{n-1} f(w_k/2).$$

Since $w_k/2 \in [z_{j+1}, z_j]$ for some $j \geq k$, we have $|f(w_k/2) - 1| \leq 2^{-k}/3$, so

$$\prod_{k=m}^{\infty} (1 - 2^{-k}/3) \leq \prod_{k=m}^{n-1} f(w_k/2) \leq \prod_{k=m}^{\infty} (1 + 2^{-k}/3).$$

Then since the infinite sum of $2^{-k}/3$ converges, the infinite products above converge, and both products approach 1 as $m \to \infty$. Thus, we can write $|h(w_m)/h(w_n) - 1| \leq \delta_m$ where $\delta_m \to 0$ as $m \to \infty$. Choose m such that $\delta_m < 1$. Since h is continuous, $h(w_m)$ is bounded for $w_m \in [z_{m+1}, z_m]$. It follows that $h(w_n)$ can be bounded independently of both n and w_m. Thus, $h(z)$ is bounded for $z \in (0, z_m]$, since every such z is equal to a w_n that satisfies the inequalities above. Also,

$$|h(w_m) - h(w_n)| \leq \delta_m |h(w_n)|.$$

Let a and $a + b$ be respectively the lim inf and the lim sup of $h(z)$ as $z \to 0^+$, both of which are finite according to the preceding paragraph. To complete the proof of the Claim, we must show that $b = 0$; if not, then $b > 0$. In that case, there exist arbitrarily small values of $z > 0$ such that $h(z) \geq a + 19b/20$. For such a z that is less than z_1, choose $m \geq 0$ such that $z \in [z_{m+2}, z_{m+1})$. Choose $w \in (0, z_m]$ such that $a - b/20 \leq h(w) \leq a + b/20$, choose $n \geq m$ such that $w \in [z_{n+1}, z_n]$, and write $w = w_n$, where w_k has the properties described above for $0 \leq k < n$. In particular, $w_m \geq z_{m+1} > z$, and

$$h(w_m) \leq h(w) + \delta_m |h(w)| \leq a + b/20 + \delta_m (|a| + |b/20|).$$

Returning to the identity (*) at the beginning of this solution, we subtract 2 from both sides and rewrite, for example, $f(z) - 1 = zh(z)$ to get

$$xf\left(\frac{z}{f(x)}\right) h\left(xf\left(\frac{z}{f(x)}\right)\right) + zh(z) = \left(x + \frac{z}{f(x)}\right) h\left(x + \frac{z}{f(x)}\right).$$

Notice that $x + z/f(x)$ goes from z to ∞ as x goes from 0 to ∞; since $w_m > z$, by continuity we can choose $x > 0$ so that $x + z/f(x) = w_m$. Then $x < w_m \leq z_m$, so $|f(x) - 1| \leq 2^{-m}/3$. Thus,

$$\left|\frac{z}{f(x)} - z\right| = \frac{|z(1 - f(x))|}{f(x)} \leq \frac{z2^{-m}/3}{1 - 2^{-m}/3} \leq z2^{-m}/2.$$

In particular, $z/f(x) \le z + z2^{-m}/2 \le 3z/2$. Since $z \le z_{m+1} = g(z_m) \le 2z_m/3$, we have $z/f(x) \le z_m$, so $|f(z/f(x)) - 1| \le 2^{-m}/3$. Next, we calculate

$$\begin{aligned} h\left(xf\left(\frac{z}{f(x)}\right)\right) &= \frac{(x+z/f(x))h(x+z/f(x)) - zh(z)}{xf(z/f(x))} \\ &\le \frac{(x+z+z2^{-m}/2)(a+b/20+\delta_m(|a|+|b/20|)) - z(a+19b/20)}{x(1-2^{-m}/3)}. \end{aligned}$$

As $m \to \infty$, the right side approaches

$$\frac{(x+z)(a+b/20) - z(a+19b/20)}{x} = a + b/20 - \frac{z}{x}(9b/10).$$

Since $z \ge z_{m+2} = g(z_{m+1}) \ge z_{m+1}/3 = g(z_m)/3 \ge z_m/9 \ge x/9$, we have $a + b/20 - (z/x)(9b/10) \le a + b/20 - b/10 = a - b/20$. Also, as z becomes arbitrarily small, so does $xf(z/f(x))$, because $x \le 9z$, and m becomes arbitrarily large. We have shown that $h(xf(z/f(x)))$ can be bounded above by a quantity that approaches $a - b/20 < a$ as $m \to \infty$. This contradicts the definition of a if $b > 0$, so we conclude that $b = 0$ and that a is the limit of $h(z)$ as $z \to 0^+$. $\square$

We can now write $f'(0) = a$, where $f'(0)$ represents the derivative from the right. Then $x + z/f(x)$ is right-differentiable at $x = 0$, with derivative $1 - zf'(0)/f(0)^2 = 1 - az$. Assume that $1 - az > 0$; then $x + z/f(x)$ is continuous and strictly increasing for $x \ge 0$ sufficiently small, with value z at $x = 0$. We next show that f is (two-sided) differentiable at z for all $z > 0$ with $1 - az > 0$. First, we subtract $f(z) + 1$ from each side of (*) and divide by $x + z/f(x) - z$ to get, for $x > 0$ sufficiently small,

$$\begin{aligned} \frac{f(x+z/f(x)) - f(z)}{x+z/f(x)-z} &= \frac{f(xf(z/f(x))) - 1}{x+z/f(x)-z} = \frac{f(xf(z/f(x))) - 1}{xf(z/f(x))}\frac{xf(z/f(x))}{x+z/f(x)-z} \\ &= \frac{f(xf(z/f(x))) - 1}{xf(z/f(x))}\frac{f(x)f(z/f(x))}{f(x)+z(1-f(x))/x} \end{aligned}$$

The right side has limit $af(z)/(1-az)$ as $x \to 0^+$, so the left side has the same limit, which is then the derivative of f from the right at z.

Next, we substitute $x = z - y$ into the functional equation from the problem statement to get

$$f((z-y)f(y)) + f(yf(z-y)) = 1 + f(z).$$

Now $(z-y)f(y)$ is right-differentiable at $y = 0$, with derivative $zf'(0) - f(0) = az - 1$. Thus, $(z-y)f(y)$ is strictly decreasing for $y \ge 0$ sufficiently small, with value z at $y = 0$. We rearrange terms in the identity above and divide by $z - (z-y)f(y)$ to get, for $y > 0$ sufficiently small,

$$\begin{aligned} \frac{f(z) - f((z-y)f(y))}{z-(z-y)f(y)} &= \frac{f(yf(z-y)) - 1}{z-(z-y)f(y)} = \frac{f(yf(z-y)) - 1}{yf(z-y)}\frac{yf(z-y)}{z-(z-y)f(y)} \\ &= \frac{f(yf(z-y)) - 1}{yf(z-y)}\frac{f(z-y)}{z(1-f(y))/y+f(y)} \end{aligned}$$

The right side has limit $af(z)/(1-az)$ as $y \to 0^+$, so the left side has the same limit, which is then the derivative of f from the left at z.

We conclude that

$$f'(z) = \frac{af(z)}{1 - az}$$

for all $z > 0$ such that $1 - az > 0$. This linear differential equation, together with the initial condition $f(0) = 1$, has a unique solution on its domain of definition; this solution can be verified to be $f(z) = 1/(1 - az)$. Then if $a > 0$, it is impossible for f to be continuous at $1/a$. Thus, we must have $a \leq 0$, in which case $1 - az > 0$ for all $z > 0$. Therefore, every solution must have the form $f(z) = 1/(1 - az)$ for all $z > 0$, for some constant $a \leq 0$. One can check that this function is in fact a solution of the functional equation in the problem statement, for each $a \leq 0$.