The 83rd William Lowell Putnam Mathematical Competition 2022

B1 Suppose that $P(x) = a_1 x + a_2 x^2 + \cdots + a_n x^n$ is a polynomial with integer coefficients, with a_1 odd. Suppose that $e^{P(x)} = b_0 + b_1 x + b_2 x^2 + \cdots$ for all x. Prove that b_k is nonzero for all $k \geq 0$.

B2 Let $\times$ represent the cross product in $\mathbb{R}^3$. For what positive integers n does there exist a set $S \subset \mathbb{R}^3$ with exactly n elements such that

$$S = \{v \times w \colon v, w \in S\}?$$

B3 Assign to each positive real number a color, either red or blue. Let D be the set of all distances $d > 0$ such that there are two points of the same color at distance d apart. Recolor the positive reals so that the numbers in D are red and the numbers not in D are blue. If we iterate this recoloring process, will we always end up with all the numbers red after a finite number of steps?

B4 Find all integers n with $n \geq 4$ for which there exists a sequence of distinct real numbers $x_1, \dots, x_n$ such that each of the sets

$$\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \dots, \{x_{n-2}, x_{n-1}, x_n\}, \{x_{n-1}, x_n, x_1\}, \text{ and } \{x_n, x_1, x_2\}$$

forms a 3-term arithmetic progression when arranged in increasing order.

B5 For $0 \leq p \leq 1/2$, let $X_1, X_2, \dots$ be independent random variables such that

$$X_i = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } p, \\ 0 & \text{with probability } 1 - 2p, \end{cases}$$

for all $i \geq 1$. Given a positive integer n and integers $b, a_1, \dots, a_n$, let $P(b, a_1, \dots, a_n)$ denote the probability that $a_1 X_1 + \cdots + a_n X_n = b$. For which values of p is it the case that

$$P(0, a_1, \dots, a_n) \geq P(b, a_1, \dots, a_n)$$

for all positive integers n and all integers $b, a_1, \dots, a_n$?

B6 Find all continuous functions $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(xf(y)) + f(yf(x)) = 1 + f(x + y)$$

for all $x, y > 0$.