Abstract Algebra

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1 Introduction

What follows is a report summarizing the proposals of a group charged with developing recommendations for undergraduate curricula in abstract algebra.[1] We begin by articulating the principles that shaped the discussions that led to these recommendations. We then indicate several learning goals; some of these address specific content areas and others address students' general development. Next, we include three sample syllabi, each tailored to meet the needs of specific types of institutions and students. Finally, we present a brief list of references including sample texts.

2 Guiding Principles

We lay out here several principles that underlie our recommendations for undergraduate Abstract Algebra courses. Although these principles are very general, we indicate some of their specific implications in the discussions of learning goals and curricula below.

Diversity of students

We believe that a course in Abstract Algebra is valuable for a wide variety of students, including mathematics majors, mathematics education majors, mathematics minors, and majors in STEM disciplines such as physics, chemistry, and computer science. Such a course is essential preparation for secondary teaching and for many doctoral programs in mathematics. Moreover, algebra can capture the imagination of students whose attraction to mathematics is primarily to structure and abstraction (for example,

[1] As with any document that is produced by a committee, there were some disagreements and compromises. The committee members had many lively and spirited communications on what undergraduate Abstract Algebra should look like for the next ten years. However, each recommendation was approved by a majority of the committee, and we have tried to arrive at a consensus on our recommendations whenever possible.

art and music majors). We note also that abstract algebra is a standard course of study at institutions of all sorts and sizes, from small liberal arts colleges to large research-intensive universities. We therefore adopt the view that our recommendations for curricula and learning goals must take into account this diversity.

Prerequisites

We think it reasonable to expect that all students taking any version of Abstract Algebra as part of a mathematics major should have studied basic linear algebra, and they should have had a course in which they were exposed to reading and writing proofs.

Non-prescription of course content

The diversity of the student community and the richness of algebra as an area of study imply a diversity of topics that could be covered in a course in Abstract Algebra. While some topic may be of prime importance for a student pursuing one goal, that topic might be relatively unimportant to a student with different goals. Moreover, some of the learning goals we articulate below can be achieved regardless of the specific concepts a given course in Abstract Algebra treats. We therefore adopt the view that there are few topics that *must* be included in any given course in Abstract Algebra. For this reason, we offer below three different syllabi, tailored to meet the needs of different institutions and students. In each case we outline what we see as some of the advantages and disadvantages of the choice of topics for various student audiences, and we do not expect any of our proposed syllabi to be followed rigidly.

Non-prescription of pedagogical techniques

Just as there is no specific set of algebraic concepts that must be covered in any given course in abstract algebra, we feel that also, no given instructor *must* adopt any specific pedagogical practice in teaching such a course. Also, courses can differ; one may give students a broad view of many algebraic ideas while another may encourage students to gain deep understandings of just a few topics. Given this, we adopt the view that we cannot recommend any specific teaching technique in preference to others.

The use of technology

We note that the use of technology in teaching and learning abstract algebra is evolving rapidly. Programs such as Gap, Magma and Sage are now being used to foster student engagement and in other ways to help students learn. We recommend that those teaching abstract algebra should carefully monitor the situation.

The importance of students' active engagement

Although we refrain from specifying pedagogical practices, we do feel that active student engagement is necessary for a mastery of algebraic ideas. In particular, it is essential that students should wrestle with hard problems and communicate their solutions with care, in writing and in speaking. In addition, problem-based, inquiry-based and collaborative learning activities are appropriate means of maintaining student engagement.

3 Cognitive Learning Goals

We articulate several specific cognitive learning goals appropriate for undergraduate courses in abstract algebra. For each of these we list some activities that may help students to achieve that goal or to demonstrate a mastery of it. We recognize, however, that given the diverse needs of various institutions and students, no specific course will address all of these goals equally.

Awareness of logical coherence Students should be aware of the logical flow of ideas in the development of the various theorems of abstract algebra. More specifically, they should understand that results depend on earlier results, and are not merely random facts. EXAMPLES: In group theory, students should see the connections between cosets, Lagrange's theorem, normality and factor groups. In ring theory, students should be aware of which facts about the ring of polynomials over a field are consequences of the division algorithm.

Understanding formal definitions Students should see the importance of precise formal definitions, including the definitions of groups, rings and fields by sets of axioms. They should appreciate the fact that these axiom systems apply to many different concrete objects. Students should also understand and be able to apply definitions of algebraic objects such as subgroups, homomorphisms and ideals, and of adjectives such as "abelian" and "normal." Students should be able to produce examples of these concepts and to solve problems concerning them. EXAMPLES: Students should be able to provide examples of a non-commutative ring, of a subring that is not an ideal, of a subgroup that is not a normal subgroup, of a homomorphism that is not an isomorphism, of a non-abelian group of order 10, *etc.*

Communication Students should be able to communicate their mathematical ideas clearly, completely, and concisely, both in writing and in speaking. EXAMPLES: Some assignments should require carefully written paragraphs, and they should be graded at least in part on the clarity and coherence of the writing. Students should have opportunities to present their ideas orally in class.

Conjectures and justifications Students should be able to recognize patterns and formulate conjectures about those patterns, and they should be able to create examples to test their conjectures, and perhaps find proofs or counterexamples. Students should also be able to read and critique the arguments of others. EXAMPLE: In the study of finite groups, but before Lagrange's theorem is introduced, students might make conjectures about possible orders of elements.

Integration and application of course concepts Students should be able to describe connections between Abstract Algebra and other mathematics courses they have taken, and they should be able to apply algebra to solve problems in other areas of mathematics and in other disciplines. Students preparing for teaching secondary school mathematics should be able to explain the links between the concepts of abstract algebra and corresponding concepts in the high school mathematics curriculum. EXAMPLES: Connections to linear algebra: Students should recognize that vector spaces are additive groups and that linear transformations are group homomorphisms. Students should see that the determinant map is a homomorphism from the group of invertible square matrices with real entries to the multiplicative group of nonzero real numbers. Connections to calculus: Students should see that polynomials over the real numbers form a ring and that differentiation is an additive group homomorphism but not a ring homomorphism. Connections to the high school curriculum: see the description of Algebra E in Section 5.

4 Other Learning Goals

The literature on learning establishes the importance of goals other than the cognitive goals with which traditional content-driven education is concerned. Such goals include helping students to build confidence at performing mathematics and encouraging them to take ownership of their own learning. Learning to persevere in the face of frustration is another important goal for students of mathematics.

5 Sample Syllabi

We present sample syllabi for three distinct undergraduate Abstract Algebra courses, which, for convenience, we call Algebra A, Algebra E and Algebra B. These are respectively: a general first course, a course for prospective high-school teachers, and a sequel to the general course suitable for students who are especially interested in algebra and for students who intend to go on to graduate school. We stress that these are only samples, and that institutions or individual instructors may wish to modify them, or to create still other syllabi that will better meet the needs of their students. We recognize that not every institution will have the resources to offer algebra courses other than the basic Algebra A, but we are providing syllabi for Algebra E and Algebra B in the hope that at least some will offer one or both of these courses. Of course, some institutions may wish to create "second" courses in Abstract Algebra different from our Algebra B, perhaps targeted at other audiences.

5.1 Algebra A

This course offers what we feel is a standard model for a first-semester Abstract Algebra course suitable for nearly every college or university. We feel some tension between the breadth of a first course that includes both groups and rings and the depth of one that focuses only on groups or only on rings. One argument for breadth is that both rings and groups are implicit in the pre-collegiate curriculum, and we feel that every student would benefit from an opportunity to see these concepts developed. For example, pre-college students encounter the rings of integers, rational numbers, real numbers, polynomials *etc.* and they will probably have also seen various groups of symmetries, both in the elementary grades and in high school geometry. We thus recommend that this one-semester course should cover both groups and rings, and also (lightly) fields. A disadvantage of this breadth is, of course, that the student has less opportunity to explore a single structure in depth. For this reason, some institutions might wish to offer an alternative first Abstract Algebra course that focuses more deeply one area: for example finite group theory. Such a course might start with definitions and examples, and eventually reach a proof of the Sylow existence theorem, and perhaps more.

The order of the topics can be chosen to suit the instructor's preference. Whichever of groups or rings is studied first, the student has the experience of learning one structure and then seeing the parallels in the second. (The analogy we make is to learning a computer language, and then the empowerment that comes with the realization of how much easier it is to learn a second language.) Our study group prefer groups first, because of their simpler definition (only one binary operation and fewer axioms) and because the familiarity of the integers and the real numbers can hide from students which statements require proof. On the other hand, we recognize that some instructors prefer to begin with rings exactly because of their familiarity to students. Studying rings first also provides some useful facts about the integers such as the division algorithm and properties of the greatest common divisor.

Suggested topics for Algebra A

Groups.

- Definitions and examples of groups and subgroups. Examples should include but not be limited to groups of rotations and reflections of planar figures and rotations of 3-dimensional objects, symmetric groups, integers modulo n with respect to addition and unit groups of integers modulo n with respect to multiplication, invertible 2×2 real matrices under multiplication.
- Cyclic groups and their subgroups, and the orders of elements.
- Symmetric groups, cycle notation, parity of a permutation and the alternating group. (See the remarks below.)
- Isomorphisms and Cayley's Theorem. (See the remarks below.)
- Cosets and Lagrange's Theorem, the falsity of the converse of Lagrange's Theorem, and if time permits, the statement of the Sylow existence theorem.
- Group actions. (See the remarks below.)
- External direct products, if time permits.
- Normal subgroups and factor groups, conjugates of a subgroup and of an element.
- Homomorphisms, and the Fundamental Homomorphism Theorem. (See the remarks below.)
- Statement of the structure theorem for finite abelian groups, if time permits.

Rings.

- Definitions and examples of rings and fields. Examples should include but not be limited to the integers and integers modulo n (including the fact that if n is prime, then one gets a field), rational, real and complex fields, polynomial rings, Gaussian integers and matrix rings. Also, if time permits, some finite fields of non-prime order can be discussed. (See the remarks below.)
- Ideals and factor rings.
- Principal ideals, integral domains, principal ideal domains, maximal and prime ideals.
- Homomorphisms, the Fundamental Homomorphism Theorem, the theorem that a commutative ring modulo a maximal ideal is a field.
- Polynomial rings and irreducible polynomials.

Remarks. Actions of groups on sets provide a modern and unifying approach to group theory, although several of us feel that this may be too abstract for many students at the level of Algebra A. Those teaching very strong students may wish to consider this approach since it can be used to construct homomorphisms from abstract groups into symmetric groups, and Cayley's theorem is an immediate consequence. This theory can also be used to construct normal subgroups. It yields, for example, the fact that a subgroup $H \subseteq G$ whose index is the smallest prime divisor of the order of G must be normal. A little deeper is the orbit-stabilizer theorem and its use in counting things, but we do not recommend covering that except perhaps for an exceptionally well-prepared student population.

Some instructors may prefer to omit the proof that the parity of a permutation is well defined. Students should understand, however, that some argument is needed; that this fact is not "obvious". We note that several proofs of this are available which avoid the tedious subscript bookkeeping required in the usual argument involving the polynomial $\prod(x_i - x_j)$.

We referred to the "Fundamental Homomorphism Theorem". By this we mean the result (in both group theory and ring theory) that the image of a homomorphism is isomorphic to the original object modulo the kernel.

We do not feel that general finite fields can be covered in this course, but it is possible to give explicit constructions of fields with 4 or 9 elements. For the field of order 9, consider objects of the form $a + bi$, where a and b are integers modulo 3 and $i^2 = -1 \equiv 2$.

Students should be exposed to applications of algebra. For example, after discussing permutation parities, the impossibility of attaining certain configurations in puzzles such as the 15-puzzle and Rubik's cube could be discussed. A good application is Polya counting, but to do that with proofs would require the orbit-stabilizer theorem. Other applications could include topics from cryptography and coding theory.

5.2 Algebra E

This one-semester course is intended for prospective high-school teachers, who would take it in place of Algebra A. Algebra E treats rings, fields and groups, with approximately two thirds of the course on rings and fields, and it highlights the connections of these topics to high school mathematics. These recommendations are aligned with those from the Conference Board of the Mathematical Sciences publication *The Mathematical Education of Teachers II*. There is a stronger argument for doing rings and fields first for this course. Some knowledge of the history of mathematics is also valuable for prospective teachers, and while we do not prescribe specific topics, we suggest that some ancillary reading could be used for this purpose.

In terms of difficulty, rigor and prerequisites, Algebra E should be comparable to Algebra A. The focus in this course, however, rests more squarely on topics that are especially relevant for future high-school teachers. In this course too, students should be exposed to some applications of algebra.

We note that other courses recommended for prospective teachers in the MET II document include topics from abstract algebra, such as number theory (*e.g.*, congruences) and geometry (*e.g.*, groups of transformations of the plane preserving distance and angles). The MET II document also recommends three courses treating high school mathematics from an advanced standpoint, and it would be natural to include connections to abstract algebra there as well.

Suggested topics for Algebra E

Rings and fields.

- Definitions and examples of rings and fields. Examples should include but not be limited to the integers, the integers modulo n, rational, real and complex fields, polynomial rings, Gaussian integers, fields of prime order and matrix rings.
- The division algorithm and its consequences for integers and for polynomial rings, including the Euclidean algorithm for both, and the remainder theorem and the factor theorem for polynomials. The fact that polynomials of degree n can have at most n roots and perhaps also the rational root theorem. (See the remarks below.)
- Ideals and factor rings.
- Principal ideals, integral domains, principal ideal domains.
- Homomorphisms, the Fundamental Homomorphism Theorem.
- The construction of the complex numbers as a quotient of the ring of real polynomials by the ideal generated by $x^2 + 1$.

Rings and fields and the high school curriculum.

- The field of complex numbers, including exponential notation, DeMoivre's theorem, and the Fundamental Theorem of Algebra (without proof).
- The arithmetic in $\mathbb{Z}$ compared with that in $\mathbb{Z}/n\mathbb{Z}$. (The goal is to understand the importance of the lack of zero divisors when solving polynomial equations by factoring.)

- The connection between the algebra of polynomial rings and the base 10 arithmetic of integers.
- The distinction between polynomials and polynomial functions.
- A brief discussion of compass and straightedge constructibility and the impossibility of trisecting an angle, squaring a circle and doubling a cube, if time permits. (Students should at least learn what those problems actually are.)

Groups.

- Definitions and examples of groups and subgroups. Examples should include but not be limited to groups of rotations of geometrical objects, symmetric groups, integers modulo n with respect to addition and unit groups of integers modulo n with respect to multiplication, invertible real 2×2 matrices under multiplication.
- Cyclic groups and orders of elements.
- Symmetric groups, the parity of a permutation and the alternating group. (But the proof that the parity is well defined can be omitted.)
- Isomorphisms. (See the remarks below.)

Some or all of the following three topics, as time permits:

- Cosets and Lagrange's Theorem and the falsity of the converse of Lagrange's theorem.
- Normal subgroups and factor groups.
- Homomorphisms and the Fundamental Homomorphism Theorem.

Groups and the high school curriculum.

- The isomorphism between the additive group of the real numbers and the multiplicative group of the positive real numbers given by the exponential and logarithm functions.
- Congruence and similarity in geometry as equivalence relations defined by groups of transformations.
- The connection between determinants and permutation parity, if time permits.

Remarks. The above list of topics seems rather full, so instructors teaching Algebra E will probably have to decide which topics are most important for their students and which to omit.

A good example of a non-obvious group isomorphism is the isomorphism between the group of rotational symmetries of a cube and the symmetric group on four symbols.

The remainder theorem for polynomials over a field is the fact that $f(a)$ is exactly the remainder when the polynomial f is divided by $x-a$, and the factor theorem is the corollary that a is a root of f precisely when $(x-a)$ is a factor. Note that the fact that a polynomial of degree n can have no more than n roots follows by induction on n, without establishing unique factorization. The rational root theorem is the fact that if a/b is a root of a polynomial f having integer coefficients, where a and b are coprime integers, then b divides the leading coefficient of f and a divides the constant term.

5.3 Algebra B

This course is a one-semester sequel to Algebra A. It is intended for students especially interested in algebra, and for students who intend to pursue graduate study in mathematics. We feel that for a course of this nature, the list of topics addressed can be somewhat more flexible than for Algebra A and Algebra E. For example, besides traditional second-semester concepts such as Sylow theory, Galois theory, advanced topics in linear algebra and perhaps modules, instructors may wish to include discussions of topics in which they have a special interest. Also, it is probably not possible to cover all of the topics listed below in one semester, so instructors will have to decide what to include and to omit. We note that there are also a number of special topics courses that can also be worthy successors to Algebra A. Without attempting to be exhaustive, we mention a few possibilities: computational algebraic geometry an introduction to representation theory, Lie groups and Lie algebras.

Suggested topics for Algebra B

Groups.

- Group actions and their orbits, the orbit-stabilizer theorem and the orbit counting formula (erroneously) attributed to Burnside and Polya counting. (See the remarks below.)
- If time permits, some of the following: conjugacy class sizes, the class equation, and the fact that nontrivial p-groups have nontrivial centers. Also the Sylow existence

theorem and the simplicity of the alternating group A_5 can be included here.

Linear algebra.

- A quick review of vector spaces, bases and dimension, linear transformations and their matrices, eigenvalues and eigenspaces.
- Minimal and characteristic polynomials and the Cayley-Hamilton theorem.
- Nilpotent linear operators and the Jordan canonical form.
- Linear functionals and dual spaces.

Fields.

- Algebraic extensions, finite degree extensions and minimal polynomials. Multiplicativity of degrees of extensions.
- Adjoining a root of a polynomial and existence of splitting fields. Repeated roots and formal derivatives. Uniqueness of splitting field. (Needed for uniqueness of finite fields.)
- Finite fields, existence and uniqueness.
- Discussion of geometric constructions. The impossibility of trisecting an angle, squaring a circle and doubling a cube. The construction of regular n-gons. (See the remarks below.)

Module theory.

- Modules, submodules, module-homomorphisms and factor modules.
- Modules over PIDs and the fundamental theorem of abelian groups. (See the remarks below.)
- The rational canonical form for linear operators.

Galois theory (brief introduction)

- Galois groups.
- Solvability by radicals. (See the remarks below.)

Remarks. The so-called Burnside orbit counting formula is $N = (1/|G|)\sum \chi(g)$, where N is the number of orbits and $\chi(g)$ is the number of fixed points of the group element g, and where the sum runs over $g \in G$. As historical research by P. M. Neumann showed, this formula is more properly attributed to Cauchy and Frobenius. An example of Polya counting is the use of this formula to count the number of essentially different ways that the faces of a cube can be colored using a palette of n colors.

At this level one cannot prove all of the relevant facts about compass and straightedge constructions. The impossibility of each the three classical hard problems comes down to the fact that a certain complex number α does not lie in a field extension of 2-power degree over the rationals. For squaring a circle, $\alpha = \sqrt{\pi}$ and for doubling a cube, $\alpha = \sqrt[3]{2}$. To see that general angle trisection is impossible, it suffices to show that a $40°$ angle cannot be constructed, and for that, we can take $\alpha = e^{2\pi/9}$, At least some part of this theory could be presented at the level of Algebra B. Also, it seems that students in Algebra B should learn about Gauss' necessary and sufficient condition that a regular n-gon can be constructed. (The condition is that n is a power of 2 times a product of distinct Fermat primes.)

There was much discussion within the working group about whether or not module theory should be included in Algebra B. One argument in favor of including it is that modules provide a natural setting for the fundamental theorem of abelian groups. This theorem can be presented, however, without the general theory of modules over PIDs, and in addition, there is a fairly easy inductive argument that can be used to show that every *finite* abelian group is a direct sum of cyclic subgroups.

As is the case with compass and straightedge constructions, it is also true that Galois theory cannot be presented with complete proofs in this course, but still, it seems appropriate for students in Algebra B to learn what a Galois group is, what it means for a group to be solvable, what it means for a polynomial to be solvable by radicals, and the connections between these ideas.

6 Resources

We list below some books that might be considered as possible texts for an undergraduate Abstract Algebra course, and we list separately books that instructors (or students) might want to consult to gain further insight into algebra, its history and its pedagogy. (Of course, the line between these two categories of books is not clearly defined.) *Please note that some of the books listed were written by the authors of this report.*

6.1 Textbooks

> Remark: *The presence of a text on this list is not meant to imply an endorsement of that text, nor is the absence of a particular text from the list meant to be an anti-endorsement. The texts are chosen to illustrate the sorts of texts that support various types of Abstract Algebra courses.*

1. Artin, Michael, *Algebra*, 2nd edition. Pearson, 2011. (A textbook for advanced undergraduates.)

2. Birkhoff, Garrett and MacLane, Saunders, *A Survey of Modern Algebra*, A. K. Peters, 1998. (This is a classic, originally published in 1941. It treats rings first.)

3. Carter, Nathan, *Visual Group Theory*, MAA, 2009. (Many diagrams, but group theory only.)

4. Cox, David; Little, John and O'Shea, Donal, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra* Springer, 2nd edition, Corrected printing, 2005. (This might be suitable for a second Abstract Algebra course different from our Algebra B.)

5. Cuoco, Al and Rotman, Joseph J, *Learning Modern Algebra: From Early Attempts to Prove Fermat's Last Theorem*, MAA, 2013. (This historically organized text is suitable for prospective teachers. It also aims to show how "important themes in algebra arose from questions related to teaching.")

6. Dubinsky, Ed and Leron, Uri, *Learning Abstract Algebra with ISETL*. Springer, 1993. (ISETL is a programming language. This text encourages students to work with algebraic ideas.)

7. Dummit, David S. and Foote, Richard M, *Abstract Algebra*, 3rd edition, John Wiley & Sons, 2003. (As with Artin's book, above, this text would be suitable only for advanced undergraduates; it may be too ambitious as a text for Algebra A.)

8. Fraleigh, John B., *A First Course in Abstract Algebra*, 7th edition, Pearson, 2003. (A very popular undergraduate algebra textbook.)

9. Gallian, Joseph A., *Contemporary Abstract Algebra*, 8th edition, Cengage, 2012. (May be the most widely used undergraduate algebra text.)

10. Gilbert, L. and Gilbert J., *Elements of Modern Algebra*, 7th edition, Brooks/Cole, 2012. (A popular textbook that is written at a more introductory level.)

11. Goodman, F. M., *Algebra: Abstract and Concrete*, Prentice Hall, 2003. (Stresses symmetry. Free online)

12. Herstein, I.N., *Topics in Algebra*, 2nd edition. John Wiley & Sons, 1975. (A classic. It might be used for a strong Algebra A class, and also for Algebra B.)

13. Herstein, I.N., *Abstract Algebra*, 3rd edtion. John Wiley & Sons, 1996. (This is probably more suitable than the older Herstein text for Algebra A.)

14. Hungerford, Thomas, *Abstract Algebra: An Introduction*, 3nd edition, Brooks/Cole, 2014. (One of the few textbooks that uses a rings-first approach.)

15. James, Gordon and Liebeck, Martin, *Representations and Characters of Groups*, Cambridge Mathematical Textbooks, Cambridge University Press, 1993. (Suitable for a "topics" course following Algebra A.)

16. Judson, Thomas W., *Abstract Algebra: Theory and Applications*, 1997 (revised 2011). (The text of the book is available free online here and available in Sage worksheet format here.)

17. Rotman, Joseph J., *A First Course in Abstract Algebra*, 3rd edition, Pearson, 2005. (This book does some commutative algebra as well as topics from linear algebra, including canonical forms.)

18. Schifrin, Theodore, *Abstract Algebra: a Geometric Approach*, Pearson, 1995. (Its genesis was a series of algebra workshops for in-service teachers. The result is a text that would be particularly suitable for Algebra E.)

19. Solomon, Ronald, *Abstract Algebra*, Pure and Applied Undergraduate Texts 9, AMS, 2003. (This text takes a novel approach, including topics not often covered in undergraduate texts. It emphasizes the historical development of the ideas and is aimed at prospective high school teachers.)

20. Stillwell, John, *Naive Lie Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, 2008. (Suitable for a "topics" course following Algebra A.)

21. Tapp, Kristopher, *Matrix Groups for Undergraduates*, Student Mathematical Library **29**, american Mathematical Society, 2005. (Suitable for a "topics" course following Algebra A.)

6.2 Other relevant books

1. Bashmakova, I.G., and Smirnova, G. S., *The Beginnings and Evolution of Algebra*, MAA, 2000.

2. Conference Board of the Mathematical Sciences. *The Mathematical Education of Teachers*, Volume 11 in the CBMS series *Issues in Education*, AMS and MAA, 2001.

3. Conference Board of the Mathematical Sciences. *The Mathematical Education of Teachers II*, Volume 17 in the CBMS series *Issues in Education*, AMS and MAA, 2012. (Available at www.cbmsweb.org.)

4. Cooke, Roger, *Classical Algebra: Its Nature, Origins, and Uses*, John Wiley & Sons, 2008.

5. Isaacs, I. Martin, *Algebra: A Graduate Course*, AMS Graduate Studies in Mathematics, 1994; originally published by Brooks/Cole. (This advanced text covers most of the topics in our syllabi thoroughly. It is a good reference for instructors and advanced students.)

6. Kleiner, Israel, *A History of Abstract Algebra*, Birkhauser, 2007.

7. Parker, Ellen Maycock, *Laboratory Experiences in Group Theory: A Manual to Be Used With Exploring Small Groups.* MAA, 1996.

8. Sultan, A. and Artzt, A., *The Mathematics that Every Secondary School Math Teacher Needs to Know*, Routledge, Taylor & Francis, 2011.

9. Usiskin, Z., Peressini, A. Ł., Marchisotto, E. and Stanley, D., *Mathematics for High-School Teachers — An Advanced Perspective*, Pearson, 2003.

10. van der Waarden, B. L., *Modern Algebra*, volumes 1 and 2, Springer, 1991. (Originally published in German in 1930. A true classic, but not suitable as an undergraduate text.)

11. Wussing, Hans, *The Genesis of the Abstract Group Concept: A Contribution to the History of the Origin of Abstract Group Theory*, Dover, 2007.