# Period Three Begins

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The logistic map

$$x_{n+1} = f(x_n) \equiv r x_n(1 - x_n), \tag{1}$$

where $x_n \in [0, 1]$ and $r \in [0, 4]$, manifests many trademark features in nonlinear dynamics, such as period-doubling and chaos [**1, 2**]. Surprisingly, all the complex behaviors can be explored by setting the parameter $r$ to different values. We focus on the particular value $r^* = 1 + \sqrt{8} \approx 3.8284$, which gives birth to the period-three cycle, where the system repeats itself every three iterations, meaning that $x_{n+3} = x_n$. One way to visualize the period-three cycle is to draw the iterating process on the cobweb plot, see FIGURE 1.

There are several different ways to derive the value $r^*$ [**3, 4, 5, 6**]. The first demonstration, given by Saha and Strogatz [**3**], unfortunately involves heavy algebraic manipulation. Later on, Bechhoefer [**4**] gives a simpler proof, where $f(f(f(x)))$ is expanded directly as a polynomial of $x$, and the coefficients are compared with their expected values. Gordon [**5**] approaches the problem by writing down the Fourier transformed version of (1) and then comparing the coefficients of different components on both sides of the equation. A more recent derivation is provided by Burm and Fishback [**6**] using Sylvester's theorem.

Here we present a new elementary derivation based on the geometry of the cobweb plot FIGURE 1.

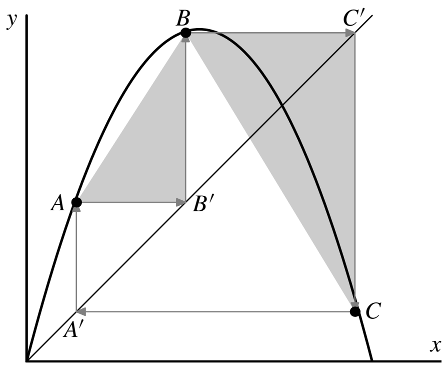


**Figure 1** The cobweb plot of the logistic map for $r = 3.84$

**The proof** Let us first denote the $x$-coordinates of $A$, $B$, and $C$ in FIGURE 1 as $a$, $b$, and $c$, respectively. The map defines a cyclic relation of $a$, $b$, and $c$: $b = f(a)$, $c = f(b)$, and $a = f(c)$. Our derivation is based on

$$\overline{A'A} + \overline{B'B} + \overline{C'C} = 0, \tag{2}$$

*Math. Mag.* **83** (2010) 295–297. doi:10.4169/002557010X521859. 

where we have used overbars to denote signed distance. Since $\overline{AB'} = \overline{A'A}$, we can express $\overline{B'B}$ as $(\overline{B'B}/\overline{AB'})\,\overline{A'A}$. The ratio $\overline{B'B}/\overline{AB'}$ can be directly calculated as $[f(b) - f(a)]/(b-a) = r(1-a-b)$. Similarly, we calculate the ratio $\overline{C'C}/\overline{BC'} = r(1-b-c)$ and replace $\overline{C'C}$ by $(\overline{C'C}/\overline{BC'})\,(\overline{B'B}/\overline{AB'})\,\overline{A'A}$. With the common factor $\overline{A'A}$ eliminated, (2) becomes

$$1 + r(1-a-b) + r^2(1-a-b)(1-b-c) = 0. \tag{3}$$

The other two symmetrical versions of (3) can be reached by cycling the symbols $a \to b, b \to c, c \to a$ twice. We now sum over the three resulting equations

$$\begin{aligned} &3 + r[3 - 2(a+b+c)] \\ &\quad + r^2[3 - 4(a+b+c) + (a+b+c)^2 + ab + bc + ca] = 0. \end{aligned} \tag{4}$$

We can further reduce (4) to an equation of a single variable $X = a+b+c$ by using the following two identities $2(ab+bc+ca) = (a+b+c)^2 - (a^2+b^2+c^2)$, and $r(a^2+b^2+c^2) = (r-1)(a+b+c)$; the latter follows from the fact $b+c+a = f(a)+f(b)+f(c)$. Thus, we have

$$r^2X^2 - (3r+1)rX + 2(1+r+r^2) = 0. \tag{5}$$

The discriminant of the quadratic is $\Delta = r^2(r^2 - 2r - 7)$. If it is positive, the equation has two roots; correspondingly, the system has two period-three cycles: a stable one and an unstable one. If it is negative, there is no root, which means that there is no period-three cycle. At the onset of period-three, the discriminant is zero, and the solution of $\Delta = 0$ gives the desired result $r^* = 1 + \sqrt{8}$. Note, the other root $r^* = 1 - \sqrt{8} \approx -1.8284$ also gives the onset of period three for the negative $r$ case, see FIGURE 2.

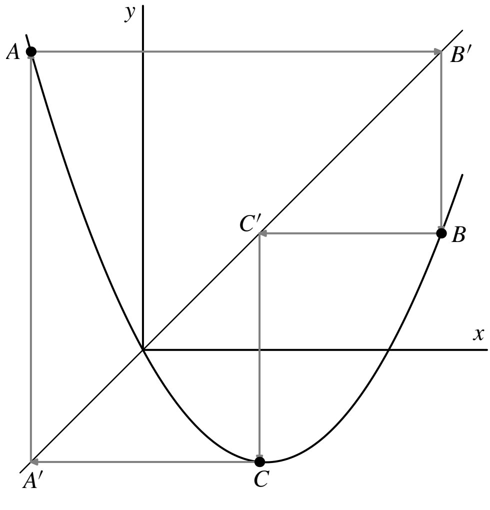


**Figure 2** The cobweb plot for $r = -1.8285$

The same derivation applies to any quadratic function $f(x)$. Actually, the algebra is much simpler if we first transform the original logistic map to $y_{n+1} = R - y_n^2$ through a change of variables, $y_n = r(x_n - 1/2)$, $R = (r^2 - 2r)/4$ [**2**]. In this case, the counterpart of (5) is $X^2 - X + 2 - R = 0$, with the discriminant being $4R - 7$. The zero-discriminant condition is readily translated to $R^* = 7/4$, or equivalently $r^* = 1 \pm \sqrt{8}$, which agrees with the previous result.

**Acknowledgment** I thank the referees for reading the manuscript carefully and for pointing out the meaning of the negative $r^*$.

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**Summary** By exploiting the geometry of the cobweb plot, we provide a simple and elementary derivation of the parameter for the period-three cycle of the logistic map.

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# Stacking Blocks and Counting Permutations

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In this paper we will explore two seemingly unrelated counting questions, both of which are answered by the same formula. In the first section, we find the surface areas of certain solids formed from unit cubes. In second section, we enumerate permutations with a specified set of restrictions. Next, we give a bijection between the faces of the solids and the set of permutations. We conclude with suggestions for further reading. First, however, it is worth explaining how this paper came about.

The author received an email from David Harris while he was helping his 12-year-old daughter Julia complete a project for her math class. Together the Harrises constructed triangular piles of cubes. After creating an increasing sequence of these piles, they computed the surface area of each pile, and hoped to find a formula for the surface area of their $n$th pile. This project and its solution are described in the next section. At the time of their correspondence, David and Julia had deduced several facts about the construction but had not yet found a formula for the surface area in general. When they searched for the first few terms in their sequence, Google returned only one hit: a Maple data file on the author's website.

The sequence that the Harrises discovered online was originally generated in the context of pattern-avoiding words and permutations. Their web search produced a conjecture that gives a nice geometric interpretation of a question about permutation patterns. This serendipitous discovery of the surprising and beautiful connection between a geometry problem and an enumeration problem illustrates how attractive new results may sometimes appear in such a surprising place as a middle-school homework exercise.

*Math. Mag.* **83** (2010) 297–302. doi:10.4169/002557010X521868. 