References

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Tennis with Markov

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In his article "A Geometric Series from Tennis" in the May 2005 issue of this *Journal*, Sandefur [**2**] showed that if the probability that player A wins a point against player B has a constant value p, then the probability that A will win a game from deuce (by the required two points) is

$$P(\text{A wins} \mid \text{deuce}) = p^2 \sum_{n=0}^{\infty} (2p(1-p))^n = \frac{p^2}{1 - 2p + 2p^2}. \qquad (*)$$

This result, established by the same method, appeared earlier in Ian Stewart's book [**3**], *Game, Set, and Math*, pp. 15–30. Furthermore Stewart gave a complete analysis not only for a single game, but for an entire best-two-of-three-sets match with possible tie-break. More recently, the result $(*)$ was illustrated using a matrix approach by Hodgson and Burke [**1**]. This method was extended to a more formal Markov chain approach by the second author in a capstone course at Washington and Jefferson College. This note describes that work.

Markov chains and stochastic matrices A Markov chain is a sequence of random values whose probabilities at the next states depend only on the state at the time, and no prior history. The controlling factor in a Markov chain is the transition probability. It is a conditional probability for the system to go to a particular new state, given the current state of the system.

Following Sandefur, we assume that p is the probability that a player wins the next point in a tennis match. This assumption means that serving is not an advantage in winning a point, as seems to be the case in women's tennis. In our tennis problem (or any win-by-2-points game), the five states that a player can reach after the deuce position are (1) loss, (2) advantage-out, (3) deuce, (4) advantage-in, and (5) win. By our assumption, the probability of making the transition between states remains constant from point to point. For instance, the probability of going from deuce to advantage-in in one service point is always p. That gives us the following transition matrix P:

$$P = \begin{array}{r} \\ \textit{from} \\ \text{Loss} \\ \text{Ad Out} \\ \text{Deuce} \\ \text{Ad In} \\ \text{Win} \end{array} \begin{array}{c} \textit{to} \\ \begin{array}{ccccc} \text{Loss} & \text{Ad Out} & \text{Deuce} & \text{Ad In} & \text{Win} \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

This is a stochastic matrix, that is, its entries are nonnegative and its row sums are all 1. Its eigenvalues and the corresponding eigenvectors can be found to be

$$\begin{array}{cccccc} & \multicolumn{5}{c}{\text{Eigenvalues}} \\ & \lambda = 1 & \lambda = 1 & \lambda = 0 & \lambda = \sqrt{2p(1-p)} & \lambda = -\sqrt{2p(1-p)} \\ \text{Eigenvectors:} & \begin{bmatrix} 1 \\ 1-p \\ 0 \\ -\frac{(1-p)^2}{p} \\ -\frac{(1-p)^2}{p^2} \end{bmatrix} & \begin{bmatrix} 0 \\ p \\ 1 \\ \frac{1-p+p^2}{p} \\ \frac{1-2p+2p^2}{p^2} \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{p-1}{p} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ \frac{\sqrt{2p(1-p)}}{p} \\ \frac{1-p}{p} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ -\frac{\sqrt{2p(1-p)}}{p} \\ \frac{1-p}{p} \\ 0 \end{bmatrix} \end{array}$$

We now show that the limit of the powers of P exists. Let M be the matrix consisting of the 5 eigenvectors of P as columns. One can check that

$$\det(M) = \frac{4(p^2 + (1-p)^2)(p-1)\sqrt{2p(1-p)}}{p^4}.$$

Since $0 < p < 1$, $\det(M) \neq 0$. Thus P has 5 linearly independent eigenvectors and so is diagonalizable. Write $P = M\Lambda M^{-1}$, where Λ is the diagonal matrix consisting of the 5 eigenvalues of P:

$$\Lambda = \operatorname{diag}\bigl(1, 1, 0, \sqrt{2p(1-p)}, -\sqrt{2p(1-p)}\bigr).$$

Since $0 < p < 1$, it is easy to check that $2p(1-p) \leq \frac{1}{2} < 1$. Thus

$$\begin{aligned} \lim_{n\to\infty} \Lambda^n &= \lim_{n\to\infty} \operatorname{diag}\left(1, 1, 0, \bigl(\sqrt{2p(1-p)}\bigr)^n, \bigl(-\sqrt{2p(1-p)}\bigr)^n\right) \\ &= \operatorname{diag}(1, 1, 0, 0, 0) = D. \end{aligned}$$

Hence $\lim_{n\to\infty} P^n = \lim_{n\to\infty} M\Lambda^n M^{-1} = MDM^{-1} = L$ exists. By straightforward computation, we find that the limit L is

$$P = \begin{array}{c} \textit{from} \\ \text{Loss} \\ \text{Ad Out} \\ \text{Deuce} \\ \text{Ad In} \\ \text{Win} \end{array} \overset{\textstyle \begin{array}{ccccc} & & \textit{to} & & \\ \text{Loss} & \text{Ad Out} & \text{Deuce} & \text{Ad In} & \text{Win} \end{array}}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 - p + \dfrac{p(p^2 - 2p + 1)}{1 - 2p + 2p^2} & 0 & 0 & 0 & \dfrac{p^3}{1 - 2p + 2p^2} \\ \dfrac{(1-p)^2}{1 - 2p + 2p^2} & 0 & 0 & 0 & \dfrac{p^2}{1 - 2p + 2p^2} \\ \dfrac{(1-p)^3}{1 - 2p + 2p^2} & 0 & 0 & 0 & \dfrac{-p(-p^2 + p - 1)}{1 - 2p + 2p^2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

The last column of L indicates the long-term forecast of winning the game at deuce at different positions. In particular,

$$L_{3,5} = \frac{p^2}{1 - 2p + 2p^2}$$

is the long-term forecast for winning a game from deuce, the result in $(*)$.

References

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Tennis (and Volleyball) Without Geometric Series

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In a recent issue of this *Journal*, Sandefur [**2**] presented an analysis of the probability of winning a deuce game in tennis (that is, any game that is tied at three points or more). Because a player must win by at least two points, this analysis led naturally to a geometric series representation of the probability that the server wins the game.

There is an alternative approach to the problem that does not rely on geometric series. Suppose that A represents a point the server wins and b represents a point that the server loses. Given that the game is at deuce, there are only four possibilities for the next two points: AA, Ab, bA, and bb. If the server wins service with probability p, these four possibilities occur with respective probabilities of p^2, $p(1-p)$, $(1-p)p$, and $(1-p)^2$. In the first case, the server wins the game; in the last case, the server loses the game; and in the other two cases, the score returns to deuce. In tennis, the scores after a 3–3 tie are frequently referred to as 'deuce' if the score is tied, 'adin' (for "advantage in") if the server is up by one point, and 'adout' if the server is down by one point, without reference to the actual score.

Let α be the probability of the server winning a game when serving from deuce. If all serves are independent,

$$\alpha = p^2 \cdot 1 + 2p(1-p) \cdot \alpha + (1-p)^2 \cdot 0.$$

Hence,

$$\alpha = \frac{p^2}{1 - 2p(1-p)}, \tag{1}$$

as obtained in [**2**]. Note that the actual score at deuce is irrelevant; given that the player is serving a game from any deuce score, the probability of subsequently winning that game is α. To obtain the overall probability that the server wins a game from the beginning, we need to multiply the probability of reaching deuce the first time by α and add this to the probability of winning without reaching deuce (as is done in [**2**]).

Of course, this approach is not all that different from the usual method of finding the sum of a geometric series. However, it has the advantage of being applicable to other more complicated situations. We present three specific examples here: the first is for determining the expected length of a game of tennis; the other two are for the probability of winning and the expected length of a game in volleyball (due to having slightly different rules, these are more complicated than for tennis).

Expected length of a tennis game The length of a sporting event may be of some interest to its organizers, as well as to broadcasters covering it. For tennis, we use the number of serves as a measure of the length of a match. To begin our analysis, we let L be the number of additional serves in a game of tennis that has reached deuce. As before, we let p denote the probability that the server wins each point, and