ways. But there are two possible patterns for the solid and dotted edges in such a cycle, so all is well. For definiteness, orient each cycle in the direction of, say, the solid edge emanating from its smallest vertex.

These bijections show that the left side of (2) is $\approx \mathcal{D}_n \times \mathcal{D}_n \times \mathcal{T}_n$ and hence $\approx \mathcal{D}_{2m} \times \mathcal{D}_{2m} \times \mathcal{T}_m \times \mathcal{T}_m$ (recall $n = 2m$). Turning to the right side of (2), we observe that there is a bijection $\mathcal{S}_{2m} \longrightarrow \mathcal{B}_{2m} \times \mathcal{S}_m \times \mathcal{S}_m$. Given $\pi \in \mathcal{S}_{2m}$, the *locations* of $1, 2, \ldots, m$ in π give an element of $\mathcal{B}_{2m}$, the *order* of $1, 2, \ldots, m$ in π gives an element of $\mathcal{S}_m$, and the order of $m+1, m+2, \ldots, 2m$ gives another. For example, $\left(\begin{smallmatrix} 1&2&3&4&5&6 \\ 4&6&3&5&1&2 \end{smallmatrix}\right)$ yields $(0\,0\,1\,0\,1\,1) \times \left(\begin{smallmatrix} 1&2&3 \\ 3&1&2 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1&2&3 \\ 1&3&2 \end{smallmatrix}\right) \in \mathcal{B}_{2m} \times \mathcal{S}_m \times \mathcal{S}_m$ where in the last permutation 1 3 2 is the rank ordering of 4 6 5. Hence the right side of (2) is $\approx \mathcal{B}_{2m} \times \mathcal{B}_{2m} \times \mathcal{S}_m \times \mathcal{S}_m$.

Thus we can identify a "square root" of each side of (2) and it now suffices to exhibit a bijection

$$\mathcal{D}_{2m}\left(\begin{smallmatrix}\text{products of}\\ \text{disjoint 2-cycles}\end{smallmatrix}\right) \times \mathcal{T}_m\left(\begin{smallmatrix}\text{unrestricted}\\ \text{0-1 sequences}\end{smallmatrix}\right) \longrightarrow \mathcal{B}_{2m}\left(\begin{smallmatrix}\text{sequences of}\\ m \text{ 0s, } m \text{ 1s}\end{smallmatrix}\right) \times \mathcal{S}_m\left(\begin{smallmatrix}\text{unrestricted}\\ \text{permutations}\end{smallmatrix}\right) \qquad (3)$$

This is quite easy: given $(\pi, \epsilon) \in \mathcal{D}_{2m} \times \mathcal{T}_m$, start with π in standard cycle form. Reverse the transpositions located in those positions where ϵ has a 1, and then arrange the transpositions in the order of their first elements. For example, with $m = 4$, $\pi = (3,2)(5,1)(6,4)(8,7)$ (in standard cycle form), and $\epsilon = (1,0,1,1)$, $(\pi, \epsilon) \to (2,3)(5,1)(4,6)(7,8) \to (2,3)(4,6)(5,1)(7,8)$. The first elements of the final product of transpositions form an m-element subset of $[2m]$ determining an element of $\mathcal{B}_{2m}$, while the rank ordering of the second elements is a permutation in $\mathcal{S}_m$. The example yields $\{2,4,5,7\} \to (0,1,0,1,1,0,1,0) \in \mathcal{B}_8$ and $(3,6,1,8) \to (2,3,1,4) \in \mathcal{S}_4$. The original pair (π, ϵ) can be uniquely retrieved, and the bijection (3) is established.

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An Easy Solution to *Mini Lights Out*

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In this MAGAZINE [**1**], Anderson and Feil demonstrated how to use linear algebra to solve the game *Lights Out*, which consists of a 5×5 array of lighted buttons; each light may be on or off. Pushing any button changes the on/off state of that light as well as the states of all its vertical and horizontal neighbors. Given a particular configuration of lights which are turned on, the object of the game is to turn out all the lights. While the computation of the solution in [**1**] is relatively straightforward, it certainly cannot be accomplished by hand in a reasonable amount of time. Analysis of a somewhat similar 3×3 game, *Merlin's Magic Square*, can be found in [**2, 3**].

Tiger Electronics has recently released a new version of the game, called *Mini Lights Out*. This consists of a 4×4 array of lighted buttons, but this time, unlike the original 5×5 version, "on" a torus. That is, the uppermost and lowermost rows are considered neighbors and likewise the leftmost and rightmost columns are considered neighbors.

Applying the techniques and notation of [**1**], we obtain not only the results predicted by Anderson and Feil, but also a complete, easy-to-compute, winning strategy for the 4×4 "mini" game.

We define the *neighborhood* of a button to be the set of buttons affected by pushing that button. That is, the neighborhood contains the button itself and its vertical and horizontal neighbors. In the original 5×5 *Lights Out* game, a neighborhood could consist of either three, four, or five buttons depending on whether the button was at a corner, on an edge, or in the interior, respectively. For the 4×4 *Mini Lights Out* game the neighborhood will always consist of exactly five buttons.

Since pushing a button twice is the same as not pushing the button at all, we can concentrate on solutions that "use" each button at most once. Also, the order the buttons are pushed does not matter. Thus we can represent any strategy by a 16×1 column vector x where each component is 1 if that button is to be pushed and 0 otherwise. In particular, the button in row i and column j is represented by the component $4(i-1)+j$.

In a similar fashion we can represent any configuration of the game by a 16×1 column vector b where each component is 1 if that button is lit and 0 otherwise.

Furthermore, any move of the game (i.e. pushing the ith button) can be represented by a 16×1 column vector v_i consisting of 1s for each button in the neighborhood and 0s elsewhere. Consider the 16×16 matrix $A = [v_1 \mid v_2 \mid \cdots \mid v_{16}]$. This matrix is most easily given by

$$A = \begin{bmatrix} B & I & O & I \\ I & B & I & O \\ O & I & B & I \\ I & O & I & B \end{bmatrix},$$

where

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \ I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

One can check that the result of applying a strategy x to an originally unlit board is simply the vector Ax. Also, due to the binary nature of the set-up, the strategy for turning off a given set of lights is identical to the strategy for turning those same lights on from an unlit board. Therefore, given an initial configuration b, our goal is to find a strategy vector x so that $Ax \equiv b \pmod 2$. This is generally done using Gauss-Jordan elimination, mod 2. However, we note that for our matrices, $B^2 \equiv I_4 \pmod 2$ and thus, $A^2 \equiv I_{16} \pmod 2$, where I_n is the $n \times n$ identity matrix. Hence, $x \equiv Ix \equiv A^2x \equiv Ab \pmod 2$ is the unique solution. Furthermore, this guarantees that every initial configuration has a winning strategy.

We note that because of the specific size of this puzzle, the neighborhoods of two distinct buttons will always intersect in either 0 or 2 buttons. Call two buttons *disjoint* if their neighborhoods are disjoint.

Define the *count* of a button to be the number of buttons in its neighborhood that are currently turned on. Furthermore, let the *parity* of a button be the parity of its count. That is, the parity of a button is 1 if an odd number of lights in its neighborhood are currently turned on, and 0 if an even number of lights are currently turned on.

Suppose a button has a current count of n. Pushing that button will change its count to $5-n$, and thus change its parity. Pushing a disjoint button will not change the original button's count or its parity. Pushing a non-disjoint button will make the original button's count either $n+2$, $n-2$, or n, depending on whether the buttons common to both neighborhoods were, respectively, originally both off, both on, or one on and one off. Thus the original button's parity will not be changed by pushing a non-disjoint button. This leads to the following result.

THEOREM 1. *The parity of a button is changed if and only if that button is pushed.*

Combining Theorem 1 and the fact that every game is winnable, we get an easily implemented method to solve the *Mini Lights Out* game. Clearly, at the end of the game each button must have parity 0. By Theorem 1, this parity can only be changed by pushing that button. Thus we merely push those buttons whose parity is one.

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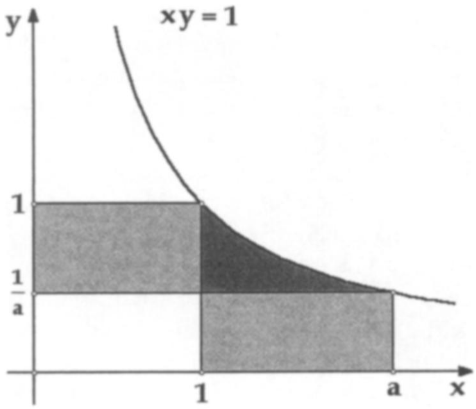
Proof Without Words: Logarithm of a Number and Its Reciprocal



$$\int_{1/a}^{1} \frac{1}{y}\,dy = \int_{1}^{a} \frac{1}{x}\,dx, \qquad a > 0$$

$$-\ln\left(\frac{1}{a}\right) = \ln a$$

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