Child's Play

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Consider the problem of finding the total number of ways each possible outcome (spot total) can be rolled using three standard dice. For the modern mathematician this problem is a fairly easy one. Nevertheless, it is scarcely child's play; the number of ways to get certain outcomes runs as high as twenty-seven. Dice are the oldest gaming instruments known to man, yet no solution to the problem was generally known until the time of Galileo. It took that great man himself to demonstrate one.

So it is perhaps bold of me to assert, as I now do, that any reasonably intelligent child, working on his (or her) own, could find all the answers to the problem in ten or fifteen minutes! He could, that is, if briefly instructed in advance as to how to proceed. As you might suspect, he would have to employ a computational tool. But it wouldn't be what you'd think. Not a computer. Not even a calculator. Rather, an apparatus more primitive than an abacus: a supply of children's toy blocks! For the sake of a better explanation, it will be assumed in what follows that the blocks are available in a variety of colors. In addition, a cardboard box would be helpful (plus, of course, a pencil and paper to record results).

The child is to stack the blocks in the simple and repetitive manner to be described, while counting the number of blocks required to complete each separate stage of construction. The numbers thus obtained represent the required solutions. That's all there is to it.

Specifically, the child, having received instruction, starts by placing a single red block in an interior corner of the box. This constitutes the first stage of construction. The count of one block indicates the number of ways a "3" can be rolled with three dice and is so recorded. There will be three faces of this first block left visible. Next, using blocks of a second color, say, blue, the child places just enough blocks atop and beside the red one to cover these faces and to leave only the blue of the second course visible. The number of blocks (three) required for this second stage indicates the number of ways a "4" can be rolled. And, again, more blocks are laid, just sufficient to cover the blue, say, with a yellow this time. Now the number of blocks required indicates the number of ways a "5" can be rolled.

And so on, color by color. It will be noted that like colors accumulate in diagonally sloping layers. Part of the instruction given will stipulate that the number of blocks in any coordinate direction is not to exceed six, i.e., no block is to be placed outside the dimensions of a $6 \times 6 \times 6$ cube (which limits can be marked on the inside of the box). However, it is by no means necessary to set up the whole cube; half is sufficient. It will be obvious that beyond the midpoint of construction the block count will be only a duplication (in reverse order) of the figures already tabulated.

When the child is through, he will have set up an interesting and ornamental structure, and will probably have had fun in the process. He will also have produced an accurate and legitimate solution to the original problem. But he won't be entitled to think he's smarter than Galileo! It's the blocks that are "smart"; they deserve the credit.

To understand how they do the job, consider first the regularity of the structure as it grows. Just as the structure of a crystal or snowflake is said to be determined by a few molecules of "seed," so here it is the initial block which establishes the final structure. Already by the time blocks of the second or third color have been added it is established that the blocks of each color will lie with their centers in a plane exclusive to that color; that all such planes will have identical slope and spacing from each other. Each of these **color planes** is normal to those interior diagonals of the blocks which extend in the general direction of added construction. The spacing between planes will be one-third of a block's diagonal (not the one-half one might intuitively expect).

The most useful coordinates for the study of the resultant structure will be the usual rectangular ones, with axes chosen parallel to the edges of the blocks, and the origin chosen so that the coordinates of the center of the first (red) block are $(1,1,1)$. This choice produces integer coordinates for the centers of all the blocks. (Each block edge is measured as 1 unit.)

To find the place of any block in the stack, we may give it an address. For instance, a given block might be the third to the right from the starting corner, the second back and the fourth up. Such an address can be abbreviated as $(3,2,4)$. But, thanks to the position chosen for the origin, the integers representing the address of any block are also the xyz coordinates which locate its center.

A fact which may be initially surprising and which underlies the counting process is: *the xyz coordinates of the centers of blocks of the same color have the same sum*. How do we know the statement is true? From analytic geometry, the general equation for a plane in rectangular coordinates is

$$Ax + By + Cz + D = 0. \tag{1}$$

But our earlier observation about the slope of the color planes says that each color plane cuts each of the coordinate axes at an equal distance from the origin. This implies that $A = B = C = 1$, so the equation of each color plane is of the form

$$x + y + z = k. \tag{2}$$

It follows, then, that the addresses for all blocks of a given color must add up to a constant peculiar to its layer. Starting with the initial red block, k is 3. Since a block in the second layer adds one unit in just one of the three coordinate directions, the sum k of its center coordinates is 4. Similarly, k for the third layer is 5, and so on.

To apply these facts to our problem of rolling three dice, we need only assign one address direction (right, back, or up) to each of the dice, and regard the integer distance of a block from the origin in this direction as the outcome of a roll of that die. Then, for each block, the total of its address coordinates will represent the outcome of a roll of the three dice. Of course, most outcomes can be reached in a plurality of ways. Our system neatly counts permutations, not combinations. It places in the stack one and only one block for each possible way a given outcome can be rolled with the three dice. Finally, the system gathers the blocks representing the ways to make a given outcome together in a layer for convenient counting. Rather a slick performance, particularly when compared with the laborious methods even a mathematical pro would have to use if proceeding with pencil and paper!

The toy-block approach is designed for the three-die problem. But we need not limit its application to dice which have the usual six faces. The number of faces on the dice enters into our solution only by limiting the expansion of the stack. This limitation is readily adjustable in three dimensions, each corresponding to the number of faces on a respective die. So even the problem of three dice with different numbers of faces—say, one with four, one with five, and one with six—could be handled with only a simple adjustment of construction boundaries.

One can't claim any practical applications for the block-counting procedure; not in this day of the computer. Nevertheless it does suggest exploration of the implications of assigning various numbers of faces to the respective three dice. Someone might even find a way to enjoy our "child's play" with tesseracts or hypercubes, thus extending the fun into one more dimension.