Choosing $Q_1 = P_1$ shows that area $\mathscr{C} = 3$ area $\mathscr{P}$ for the special polygonal cycloid, and letting $n \to \infty$ shows that the area of an arch of a smooth cycloid is three times the area of the circle used to generate it.

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Maybe the Price Doesn't Have to be Right: Analysis of a Popular TV Game Show

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Television game shows provide excellent classroom examples for teaching elementary concepts of probability. In this capsule, we will use elementary probability concepts to determine the probability of winning a game that has appeared on the popular TV show: *The Price is Right*. The results are rather surprising, and we invite the reader to intuitively speculate on the final result.

Consider the following game. A small ball is hidden under one of four shells resting on a table (Figure 1). The contestant will win the grand prize if he identifies the shell that covers the ball. However, in order to work in some product advertising, the following preliminary activity is incorporated into the game. Four products are placed before the contestant. Listed with each product is a dollar amount that is *not* the correct price of the item, as illustrated in Figure 2.

Figure 1. A ball is hidden under one of four identical shells.

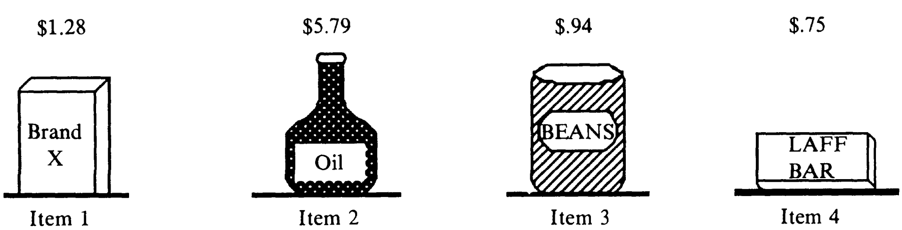

Figure 2. Four products and corresponding incorrect prices.

For each of the items, the contestant is asked to tell whether the actual price is higher or lower than the price shown for the item. The total number of correct predictions is recorded and then the contestant is allowed to randomly select this number of shells, winning the grand prize if the hidden ball is uncovered. The basic question is:

What is the probability that the contestant will win the grand prize?

To answer this question, we need to describe the problem in mathematical terms and clearly state our assumptions. Let p_i denote the probability of the event that the contestant makes the correct decision for item $i (i = 1, 2, 3, 4)$ and let $q_i = 1 - p_i$. If B denotes the event that the shell with the ball is selected, then we desire $P(B)$, the unconditional probability of event B. Before proceeding, the reader is invited to speculate about the form of the final answer for $P(B)$. Where does your intuition lead?

To aid our intuition, let's consider the special case where $p_1 = p_2 = p_3 = p_4 = p$. In this case, assuming independence of the decisions, it is easy to see that the number X of correct price predictions has a binomial probability distribution with parameters $n = 4$ and p. Since the mean value of X is $np = 4p$, we might reason informally and guess that $P(B)$ equals the average number of correct predictions (the number of shells the player gets to choose) divided by 4; that is, $4p/4 = p$. Is this valid reasoning? If so, how is it extended to the general case?

To continue our intuitive approach and answer the latter question, define I_i to be 1 if a correct decision is made on item i, and let it be 0 otherwise. Then $X = \sum_{i=1}^{4} I_i$ and, in this general case, the expected value of X is

$$E(X) = E\left(\sum_{i=1}^{4} I_i\right) = \sum_{i=1}^{4} E(I_i) = \sum_{i=1}^{4} p_i = p_1 + p_2 + p_3 + p_4. \tag{1}$$

Note that independence of $I_i (i = 1, 2, 3, 4)$ is *not* used here. Reasoning intuitively, as in the special case, we speculate that $P(B)$ equals the expected number of correct decisions divided by 4, or $(p_1 + p_2 + p_3 + p_4)/4$. This reduces to our former guess when the p_i's have the common value p.

Now let's provide a rigorous derivation for $P(B)$, using only fundamental rules of probability. We begin by expressing event B as the union of disjoint events $B = \bigcup_{k=0}^{4}\{B,\ X = k\}$, where $\{B,\ X = k\}$ denotes the intersection of the events B and $\{X = k\}$. Recall (product rule) that $P(F \cap G) = P(F|G)P(G)$, where $P(F|G)$ is the conditional probability of event F, given event G. Therefore,

$$\begin{aligned} P(B) &= P\left(\bigcup_{k=0}^{4} \{B, X = k\}\right) = \sum_{k=0}^{4} P(\{B, X = k\}) \\ &= \sum_{k=0}^{4} P(B|X = k)P(X = k). \end{aligned} \tag{2}$$

But from the definition of the game,

$$P(B|X = k) = k/4 \qquad (k = 0, 1, 2, 3, 4). \tag{3}$$

Substituting (3) into (2), and using the definition

$$\sum_{k=0}^{4} kP(X = k) = E(X),$$

we obtain

$$P(B) = \sum_{k=0}^{4} (k/4)P(X = k) = (1/4)\sum_{k=0}^{4} kP(X = k) = E(X)/4.$$

Therefore, from (1),

$$P(B) = (p_1 + p_2 + p_3 + p_4)/4, \tag{4}$$

which confirms our informal derivation!

If (1) is not used, the quantity $P(X = k)$ must be computed. Assuming independence of the decisions and again using the idea of decomposition into the union of disjoint events, we obtain

$$P(X = 0) = q_1 q_2 q_3 q_4$$

$$P(X = 1) = p_1 q_2 q_3 q_4 + q_1 p_2 q_3 q_4 + q_1 q_2 p_3 q_4 + q_1 q_2 q_3 p_4$$

$P(X=2)=p_1p_2q_3q_4+q_1p_2p_3q_4+q_1q_2p_3p_4+p_1q_2p_3q_4+p_1q_2q_3p_4+q_1p_2q_3p_4$

$P(X=3)=p_1p_2p_3q_4+p_1p_2q_3p_4+p_1q_2p_3p_4+q_1p_2p_3p_4$

$P(X=4)=p_1p_2p_3p_4.$

Substituting the expressions for $P(X=k)$ into $\Sigma_{k=0}^4 kP(X=k)$ will yield (4).

In the best case, our contestant has perfect knowledge of product prices, so $p_1=p_2=p_3=p_4=1$ and $P(B)=1$. On the other hand, a contestant who makes pure guesses has $p_1=p_2=p_3=p_4=1/2$, yielding $P(B)=1/2$. Thus, assuming a contestant does not purposefully attempt to make an incorrect price decision, we have the bounds

$$1/2 \le P(B) \le 1.$$

This result is rather surprising since people intuitively believe that when the items are unfamiliar to the contestant, $P(B)$ will be below $1/2$. Our results show that this does not occur since a player can always achieve $p_i \ge 1/2$, and $P(B)$ is monotonically increasing in the p_i's. A strategy for achieving $p_i \ge 1/2$ is for the player to make a random guess except when he is sure of the correct price. In an experiment we performed involving 69 subjects (student "contestants"), 35 subjects (roughly 51%) won the game.

We wonder if these results were known by the game's designers.

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Amalgamation of Formulae for Sequences

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Many problems involve formulae that list cases for the different residue classes modulo some integer r. For example, let $L_n=\{1,3,4,7,\dots\}$ and $f_n=\{1,1,2,3,\dots\}$ denote the usual Lucas and Fibonacci numbers, respectively. Then

$$L_n^2=\begin{cases}5f_n^2+4, & \text{if } n\equiv 0 \pmod 2\\ 5f_n^2-4, & \text{if } n\equiv 1 \pmod 2\end{cases}.$$

In this example, the two formulae (one for each case) can be amalgamated into the single formula

$$L_n^2=5f_n^2+(-1)^n4.$$

In this capsule, we describe how to amalgamate a set of k formulae

$$u_n=\alpha_r(n) \quad \text{for } n\equiv r \pmod k \qquad (\text{where } 0\le r\le k-1)$$

into a single formula.

Motivation. In 1964, the author and a colleague, A. L. Dulmage, were interested in the following problem: Let $a_1, a_2, \dots, a_r$ be a set of positive integers having g.c.d. = 1. Then there is a largest nonnegative integer, denoted $F(a_1, a_2, \dots, a_r)$, that cannot be expressed in the form $c_1a_1+c_2a_2+\cdots+c_ra_r$ for nonnegative integers $c_1, c_2, \dots, c_r$. The function F has been the object of much study. The case $r=2$ is well known, the result being $F(a_1,a_2)=a_1a_2-a_1-a_2$. For $r=3$, S. M. Johnson ["A Linear Diophantine Problem," *Canad. J. Math.* 12 (1960) 390–398] gave an efficient algorithm for calculating $F(a_1,a_2,a_3)$ but did not obtain an explicit formula. In "Gaps in the Exponent Set of Primitive Matrices," *Illinois J. Math.* 8 (1964) 642–656, A. L. Dulmage and N. S. Mendelsohn described a graphical method that, in principle, allows one to solve explicitly for many classes of