Also, the relation in (4) implies that $\sum_{r=0}^{n} \binom{n}{r} H_r^n = 1$.

References

[1] M. Bicknell-Johnson, Diagonal sums in the harmonic triangle, The Fibonacci Quarterly, 19 (1981) 196–198.
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A Note on Evaluating Limits Using Riemann Sums

SUDHIR K. GOEL
DENNIS M. RODRIGUEZ
University of Houston-Downtown
Houston, Texas 77002

In this note, we present a method for evaluating some limits. This method, which does not appear to be widely used, involves expressing the limit in the form of a Riemann sum. This sum, in turn, is equivalent to a definite integral, which can be evaluated using standard techniques. We illustrate the technique with several examples.

The first example appears as an exercise in the book *Advanced Calculus* by David V. Widder [**2**, p. 391]. It can also be solved using Stirling's formula or series methods.

Example 1. Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{n}\ln n! - \ln n\right).$$

Solution. We have

$$\begin{aligned}\lim_{n\to\infty}&\left(\frac{1}{n}\ln n! - \ln n\right)\\ &= \lim_{n\to\infty}\left[\frac{1}{n}\left(\sum_{k=1}^{n}\ln k\right) - \ln n\right]\\ &= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{k=1}^{n}(\ln k - \ln n)\right]\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right).\end{aligned}$$

This limit is the Riemann sum of $f(x) = \ln x$ over the interval $[0,1]$. Hence,

$$\lim_{n\to\infty}\left(\frac{1}{n}\ln n! - \ln n\right) = \int_0^1 \ln x\,dx.$$

Integrating this improper integral by parts, and using l'Hôpital's rule, we get

$$\int_0^1 \ln x\,dx = -1.$$

Hence,

$$\lim_{n\to\infty}\left(\frac{1}{n}\ln n! - \ln n\right) = -1.$$

Our next example is a problem which was proposed by Norman Schaumberger in the March 1984 issue of *The College Mathematics Journal* [3].

Example 2. Evaluate

$$\lim_{n\to\infty}\frac{1}{n}[2\cdot 5\cdot 8\cdots(3n-1)]^{1/n}.$$

Solution. Let

$$y = \frac{1}{n}[2\cdot 5\cdot 8\cdots(3n-1)]^{1/n}.$$

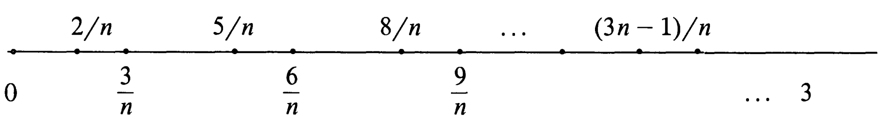
Taking ln of both sides, we obtain

$$\begin{aligned}\ln y &= \frac{1}{n}\left[\sum_{k=1}^{n}\ln(3k-1)\right] - \ln n\\ &= \frac{1}{n}\sum_{k=1}^{n}[\ln(3k-1) - \ln n]\\ &= \frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{3k-1}{n}\right).\end{aligned}$$

Taking the limit as $n\to\infty$ of both sides, we obtain

$$\begin{aligned}\lim_{n\to\infty}\ln y &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{3k-1}{n}\right)\\ &= \frac{1}{3}\lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\ln\left(\frac{3k-1}{n}\right).\end{aligned} \tag{A}$$

The last expression in (A) above is a Riemann sum for the function $f(x) = \ln x$ over the interval $[0,3]$. This is clear with the help of the following figure.



Thus,

$$\ln\left(\lim_{n\to\infty} y\right) = \frac{1}{3}\int_0^3 \ln x\,dx$$

Integrating this improper integral by parts, and using l'Hôpital's rule, we get

$$\ln\left(\lim_{n\to\infty} y\right) = \frac{1}{3}(3\ln 3 - 3) = \ln 3 - 1.$$

Thus,

$$\lim_{n\to\infty} y = e^{(\ln 3 - 1)} = \frac{3}{e}.$$

Hence,

$$\lim_{n\to\infty}\frac{1}{n}[2\cdot 5\cdot 8\cdots(3n-1)]^{1/n} = \frac{3}{e}.$$

Note. Example 2 above can be generalized as follows:

$$\lim_{n\to\infty}\frac{1}{n}[(k-1)(2k-1)(3k-1)\cdots(kn-1)]^{1/n}=\frac{k}{e}\quad\text{for } k=2,3,4,\ldots.$$

We tried to prove Stirling's formula using the above technique but we were not successful. However, we were able to prove the following result which is very similar to Stirling's formula.

Example 3.

$$\lim_{n\to\infty}\left(\frac{n!e^n}{n^{n+(1/2)}\sqrt{2\pi}}\right)^{1/n}=1.$$

Solution. Let

$$y=\left(\frac{n!e^n}{n^{n+(1/2)}\sqrt{2\pi}}\right)^{1/n}.$$

Taking the natural log of both sides, we obtain

$$\ln y=\frac{1}{n}\left[\ln n!+n\ln e-\left(n+\frac{1}{2}\right)\ln n-\frac{1}{2}\ln 2\pi\right]$$

$$=\frac{1}{n}\ln n!+1-\ln n-\frac{\ln n}{2n}-\frac{\ln 2\pi}{2n}$$

Taking the limit as $n\to\infty$ of both sides, we obtain

$$\lim_{n\to\infty}\ln y=\lim_{n\to\infty}\left(\frac{1}{n}\ln n!-\ln n\right)+1$$

$$=-1+1\text{ (using example 1)}$$

$$=0.$$

Hence,

$$\lim_{n\to\infty}y=e^0=1.$$

We now illustrate how this technique can be used in approximating partial sums of divergent p-series.

Example 4. We know that the p-series $\sum_{k=1}^{\infty}k^p$ converges if $p<-1$ and diverges otherwise. In this example, we show that for large n and $p>-1$, $\sum_{k=1}^{n}k^p$ is approximately $n^{p+1}/(p+1)$ in the sense that the limit of the ratio of the two quantities as $n\to\infty$ is 1. We have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^p}{n^{p+1}}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^p$$

$$=\int_0^1 x^p\,dx$$

$$=\left(\frac{x^{p+1}}{p+1}\right)\Bigg|_0^1$$

$$=\frac{1}{p+1}\quad\text{for } p>-1.$$

Hence for large n and $p>-1$, we have $\sum_{k=1}^{n}k^p$ is approximately $n^{p+1}/(p+1)$. We observe that $n^{p+1}/(p+1)$ is the first term in the approximation of p-series by Riemann-Zeta functions [4]; however, this requires sophisticated machinery in comparison to the method described above.

With the aid of a computer, we calculated the value of $\sum_{k=1}^{n} k^p$ and $n^{p+1}/(p+1)$ for various values of n and p. Our results (summarized in TABLES 1 and 2 below) show that the percent error is small.

TABLE 1

n	p	$\sum_{k=1}^{n} k^p$	$\frac{n^{p+1}}{p+1}$	% Error
100	1	5050	5000	.99
100	2	338350	333333.33	1.48
100	3	25502500	25000000	1.97
100	4	2050333330	2000000000	2.45
200	1	20100	20000	.50
200	2	2686700	2666666.67	.75
200	3	404010000	400000000	.99
200	4	64802666660	64000000000	1.24
400	1	80200	80000	.25
400	2	21413400	21333333.33	.37
400	3	6432040000	6400000000	.49
400	4	2060821333320	2048000000000	.62

Notice that for fixed p, the % error decreases as n increases. Computer results for large values of n and p are given in TABLE 2 below.

TABLE 2

n	p	% Error
300	14	2.46
500	13	1.39
700	12	.92
800	12	.81
1000	11	.60

There exist many other limits that can be evaluated using Riemann sums, three of which are listed below.

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}. \tag{1}$$

$$\lim_{n\to\infty} \sum_{k=n+1}^{rn} \frac{1}{k} = \ln r \quad \text{for } r = 2, 3, 4, \dots . \tag{2}$$

$$\lim_{n\to\infty} \sum_{k=np}^{nq} \frac{1}{k} = \ln\left(\frac{q}{p}\right) \text{ for positive integers } p \text{ and } q \text{ with } p < q. \tag{3}$$

Limit (1) above appears in most advanced calculus texts and is usually solved either by Stirling's formula or by series methods. Limits (2) and (3) above appear as exercises in Widder's Advanced Calculus Book [**2**, p. 392] in the section dealing with Stirling's formula.

References

[1] Michael Spivak, Calculus, 2nd ed., Publish or Perish Inc., 1980.
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