$$R + QR + Q^2R + Q^3R + \dots = (I_t + Q + Q^2 + Q^3 + \dots)R.$$

It can be shown that all the entries of Q^n approach zero as n tends to infinity (see [**2**, pp. 43–45]). This condition yields the following matrix generalization of the familiar formula for the sum of a geometric series:

$$(I_t - Q)^{-1} = I + Q + Q^2 + Q^3 + \dots$$

(see [**2**, p. 22]). Thus we see that the (i, j)-entry of the $t \times s$ matrix $(I_t - Q)^{-1}R$ gives the probability that the Markov chain ends up in absorbing state j given that the initial state was $s + i$.

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Poker With Wild Cards—A Paradox?

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I participate in a sporadic poker game whose organizer detests any use of wild cards. (A *wild card* can be called anything its holder wishes.) I'd always attributed this aversion to some personality quirk. Then I discovered a reason to share his concern.

After a recent class in which I tossed out an unsubstantiated claim about wild cards sometimes altering the accepted hierarchy of poker hands, I decided I'd better actually do the calculations before my students did. I wasn't surprised to substantiate my claim, but I was surprised to discover that unresolvable inconsistencies can arise when wild cards are used. This note shows how, in one common situation, *no matter what hierarchy is established, the resulting probabilities are incompatible with it.* So perhaps my friend (who happens to be a political scientist, as well as the frequent victor in our always-friendly games) has more innate mathematical talent than either of us realized.

The usual hierarchy of poker hands (when played without wild cards) is, from best to worst, royal flush, straight flush, four-of-a-kind, full house, flush, straight, three-of-a-kind, two pair, one pair, and junk.[1] Without wild cards, this hierarchy is consistent

[1]Some of these terms may not be self-explanatory. A *royal flush* consists of an ace, king, queen, jack, and ten, all in one suit. A *straight flush* comprises five in a row, all in one suit (but not a royal flush). A *full house* includes three-of-a-kind and one pair. A *flush* consists of five cards in one suit (but not a royal flush or a straight flush). A *straight* has five in a row (but not a royal flush or a straight flush). Any other hand is *junk*.

with the relative frequency of the hands. (For the calculations for 5 card poker without wild cards, see **[2]**. Packel allows an ace to be either high or low in a straight (instead of just high). This does not affect the hierarchy itself.)

If the two jokers are added as wild cards, one more hand is possible: five-of-a-kind. The first table gives frequencies and probabilities for all possible hands. The verifications of these frequencies are nice exercises in combinatorics. Here's one sample of the systematic (if somewhat pedantic) reasoning involved. For four-of-a-kind, there are three distinct ways to specify the hand without redundancy.

(a) Select no joker of the two; select one denomination from the thirteen; select all four of that denomination; select one denomination from the remaining twelve; and select one of the four of that denomination.
(b) Select one joker of the two; select one denomination from the thirteen; select three of the four of that denomination; select one denomination from the remaining twelve; and select one of the four of that denomination.
(c) Select two jokers of the two; select one denomination from the thirteen; select two of the four of that denomination; select one denomination from the remaining twelve; and select one of the four of that denomination.

Thus the number of ways to get four-of-a-kind is

$$\binom{2}{0}\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}+\binom{2}{1}\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{1}+\binom{2}{2}\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{1}$$
$$= 624 + 4992 + 3744 = 9360.$$

TABLE 1 Wild card poker frequencies and probabilities, based on the usual hierarchy

Rank	Type	Frequency	Probability
1	FIVE-OF-A-KIND	78	0.000025
2	ROYAL FLUSH	84	0.000027
3	STRAIGHT FLUSH	480	0.000152
4	FOUR-OF-A-KIND	9360	0.002960
5	FULL HOUSE	9360	0.002960
6	FLUSH	11448	0.003620
7	STRAIGHT	30540	0.009657
8	THREE-OF-A-KIND	233584	0.073860
9	TWO PAIR	123552	0.039068
10	ONE PAIR	1440464	0.455481
11	JUNK	1303560	0.412192
	(TOTAL)	$\binom{54}{5} = 3162510$	1

Observe one anomaly in the first table: Three-of-a-kind and two pair are in the wrong order. But if their positions are reversed, many hands that would have been three-of-a-kind are now best called two pair. For example, {A♣, 8♥, 4♠, JOKER, JOKER} can be called "three aces" (if three-of-a-kind beats two pair) or two aces and two eights (if two pair beats three-of-a-kind). So the numbers of these two types of hands change, as shown in the second table: Two pair and three-of-a-kind are in the wrong order again! (In fact, the situation is relatively worse than before the reversal.)

A look at the first table reveals that the same phenomenon occurs with one pair and junk. For example, {A♣, 8♥, 4♠, 2♦, JOKER} can be called "two aces" (if one pair beats junk) or "junk" (if junk beats one pair, calling the JOKER a king, say). (For other situations in which junk beats one pair or even two pair, see **[1]**.)

TABLE 2 Wild card poker frequencies and probabilities, based on a revised hierarchy

Rank	Type	Frequency	Probability
1	FIVE-OF-A-KIND	78	0.000025
2	ROYAL FLUSH	84	0.000027
3	STRAIGHT FLUSH	480	0.000152
4	FOUR-OF-A-KIND	9360	0.002960
5	FULL HOUSE	9360	0.002960
6	FLUSH	11448	0.003620
7	STRAIGHT	30540	0.009657
8	TWO PAIR	302224	0.095565
9	THREE-OF-A-KIND	54912	0.017363
10	JUNK	1645784	0.520404
11	ONE PAIR	1098240	0.347268

The more one looks, the worse it gets. In the original hierarchy, there were 9360 four-of-a-kind hands and 9360 full house hands. So one could arbitrarily decide to rank a full house above four-of-a-kind. But this would really be disastrous, for then there would turn out to be 18096 full houses and 624 four-of-a-kind! With two added jokers as wild cards, there is *no* hierarchy of hands that is consistent with the frequency of the hands.

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Counting Squares in $\mathbb{Z}_n$

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An elementary number theory problem is to determine the possible forms of squares among the positive integers. For instance, it is easy to see that any square must be of the form $3k$ or $3k+1$. (Since every positive integer can be written as either $3q$, $3q+1$, or $3q+2$, simply square these numbers and simplify.) Restated, this assertion is that 0 and 1 are the squares in $\mathbb{Z}_3$, the ring of equivalence classes of integers modulo 3. In general, a square has the form $nk+r$ if, and only if, r is a square in the ring $\mathbb{Z}_n$. How many squares are there in $\mathbb{Z}_n$?

Fundamental notions An element a in $\mathbb{Z}_n$ is a *square* in $\mathbb{Z}_n$ if and only if $x^2 = a$ has a solution in $\mathbb{Z}_n$. The *units* of $\mathbb{Z}_n$ are the elements that are relatively prime to n. The units that are squares are commonly called *quadratic residues* (or, more precisely, the quadratic residues mod n in a reduced residue system) [**1**, p. 84]. The quadratic residues have been completely characterized [**2**, p. 201], and the standard results will be utilized in what follows.