Since $f(K)$ is a product of positive, decreasing functions of t, it is itself a positive, decreasing function of t. Since $\alpha \geq \pi/3$, we have $t \geq \sqrt{3}$. Hence the maximal value of $f(K)$ is attained when $t = \sqrt{3}$, that is, when T is an equilateral triangle. In this case

$$f(K) = \frac{1}{w}(2R - D) \leq \frac{2}{3}(2 - \sqrt{3}).$$

Now suppose that K has no lattice point in its interior. Combining (4) with (2) gives

$$2R - D \leq \frac{1}{3},$$

with equality when and only when K is congruent to the equilateral triangle shown in Figure 1. ∎

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Cutting a Polyomino into Triangles of Equal Areas

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In 1970 Monsky proved that a square cannot be cut into an odd number of triangles of equal areas [**1**], [**6**, p. 118]. This result has been generalized four times. Mead proved that when an n-dimensional cube is cut into simplices of equal volumes, the number of simplices is a multiple of $n!$ [**2**]. Kasimatis proved that when a regular n-gon, $n \geq 5$, is cut into triangles of equal areas, the number of triangles is a multiple of n [**3**]. Stein proved that the theorem about the square holds for any centrally symmetric polygon with at most eight sides [**4**] and Monsky generalized this to any centrally symmetric polygon [**5**]. In this note we extend the theorem about squares to polyominoes consisting of an odd number of squares.

By a *standard square* in the *xy*-plane we mean a unit square whose corners have integer coordinates. A *standard segment* is a line segment of unit length joining two points with integer coordinates. A *polyomino* is the union of a finite number of standard squares.

Conjecture 1. When a polyomino is cut into triangles of equal areas, the number of triangles is even.

That is a special case of

Conjecture 2. When a polygon in the *xy*-plane that is bounded by lines parallel to the axes is cut into triangles of equal areas, the number of triangles is even.

The following theorem confirms a special case of the first conjecture.

Theorem. *When a polyomino consisting of an odd number of standard squares is cut into triangles of equal areas, the number of triangles is even.*

We use the machinery described in [**1**] and [**6**, pp. 110–117], which we summarize briefly. Define $\varphi : Q \to Q$ by $\varphi(2^a b/c) = a$, where b and c are odd, and $\varphi(0) = \infty$. For instance, $\varphi(2) = 1$, $\varphi(3) = 0$, and $\varphi(5/2) = -1$. Label a point $(x, y) \in Q \times Q$ by P_0 if $\varphi(x) > 0$ and $\varphi(y) > 0$, by P_1 if $\varphi(x) \leq 0$ and $\varphi(y) \geq \varphi(x)$, and by P_2 if $\varphi(x) > \varphi(y)$ and $\varphi(y) \leq 0$. For example, $(2, 0)$ is labeled P_0, $(1, 3)$ is labeled P_1, and $(2, 1)$ and $(1, 1/2)$ are labeled P_2. It can be shown that if a line segment formed of standard segments has ends labeled P_1 and P_2, then the ends of the individual segments are labeled either P_1 or P_2 and an odd number of them have both labels. The following lemma [**6**, p. 118] is the key tool in establishing the theorem.

Lemma. *Let a polyomino R have area A. Assume that on the boundary of R are an odd number of standard edges with ends labeled P_1 and P_2. Then $\varphi(n) \geq \varphi(2A)$ if R is cut into n triangles of equal areas.*

Proof of the theorem: As may be checked, the only standard segments whose ends are labeled P_1 and P_2 are parallel to the x-axis and lie on lines with an odd y-coordinate. Thus on the border of each standard square is one edge with the labels P_1 and P_2. Edges in the interior of R are adjacent to two standard squares, while edges on the boundary are adjacent to one standard square in R. Since there are an odd number of standard squares in R, the assumption of the lemma holds. Because A is an integer, $\varphi(2A) \geq 1$. Thus the number of triangles is even. ■

The theorem also holds for polyominoes consisting of at most 6 standard squares. (Incidentally, the even case implies the odd one since each standard square can be cut into four congruent squares.) As an illustration, consider the polyomino of area 6 formed of a row of four standard squares with a square attached at each end on the same side of the row, as shown in Figure 1. Note that it can be cut into six triangles of equal areas. No matter how we rotate or translate the polyomino, the hypothesis of the lemma cannot apply, for the conclusion would

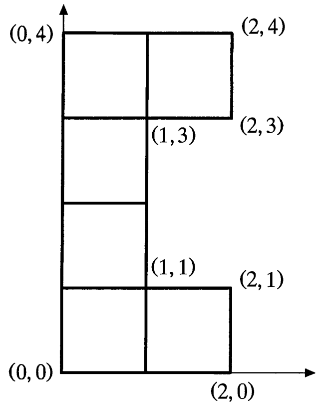

Figure 1

be that the number of triangles would be a multiple of 4. It is necessary to transform the polyomino so that the image has an area A for which $\varphi(A) \leq 0$. Applying the transformation $(x, y) \to (x, y/2)$ followed by the translation by $(1, 1)$ produces a labeling described in the hypothesis of the lemma, as may be verified.

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A Short Proof of Turán's Theorem

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Extremal graph theory is the search for the thresholds in edge density where substructures of interest are forced to appear in graphs. The canonical extremal theorem involving structure S is of the type: If G is a graph with n vertices containing no S, then G has no more than $f(n)$ edges. The genesis moment of extremal graph theory occurred in 1941 with Turán's article [**1**] in which he proved the canonical extremal result for $S = K_r$, a complete graph with r vertices. The purpose of this note is to provide a new and perhaps shorter proof than has previously been noticed.

Theorem (Turán, 1941). *Graphs with n vertices containing no K_r have no more than $(r-2)n^2/(2r-2)$ edges , for $r \geq 2$.*