r_c', r_c decreases on some interval $[0, \epsilon]$ and thus $r_c(\epsilon) < r$. On the other hand, under the assumptions listed in (4), the condition $0 \le r \le D$ is sufficient but not necessary for the conclusion that for *some* values of $p \in (0, 1)$, $r_c > r$. For example, for the generalized FPP Problem with $p = q = 1/2$ and thus $\lambda = 1/4$, the right side of (3) is $4/(81)[7 + \sqrt{130}] \doteq .909$. Thus if $(56/65) < r < 4/(81)[7 + \sqrt{130}]$, then $r_c(1/4) > r$ even though $r > D$.

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On the Remainder in the Taylor Theorem

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We give a short straightforward proof for a bound on the reminder term in the Taylor theorem. The proof uses only induction and the fact that $f' \geq 0$ implies the monotonicity of f, so it might be an attractive proof to give to undergraduate students.

Let f be an n-times differentiable function in a neighborhood of $a \in \mathbb{R}$. Recall that the Taylor polynomial of order n of f at a is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

It will be convenient to define $P_{-1}(x) = 0$. Let $R_n = f - P_n$ be the remainder term. Then

Theorem 1 (Lagrange's formula for the remainder). *If f has an $(n+1)$th derivative on $[a, b]$, then there exists $\xi \in [a, b]$ such that*

$$R_n(b) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}.$$

This formula is the main tool for bounding the remainder of the Taylor expansion in calculus classes, especially when this subject is taught before integration. One would like to have some "natural" proof for it. In [**3**] it is suggested that induction seems suitable, since P_n' is the Taylor polynomial of f' of order $n-1$, so $R_n'(x)$ is given by induction. This approach fails, since one cannot integrate $R_n'(x)$ because the point $\xi = \xi(x)$ depends on x.

While we were teaching a first calculus course for chemistry and physics majors at Tel-Aviv University, we observed that this obstacle can be removed if we change the problem to finding a **bound** on the remainder. This is just as useful, since a bound is all that is needed to show that the Taylor series converges to the function. From our personal experience, we believe that this approach enables students to grasp the material more easily. Furthermore, Lagrange's formula can be deduced from the bound, as we show at the end of this note.

The only fact needed in the proof is that a function with a positive derivative is increasing. This can be proved easily with the mean value theorem or without it (see [**1, 2**]). As a direct corollary one gets:

Lemma 2. *Let f, g be differentiable in a closed segment $[a, b]$. If $f(a) = g(a)$ and $f'(x) \leq g'(x)$ for every $x \in (a, b)$, then $f(x) \leq g(x)$ for every $a \leq x \leq b$.*

Theorem 3. *Suppose that f has an $(n+1)$th derivative on $[a, b]$ and that $m \leq f^{(n+1)}(x) \leq M$ for every $x \in [a, b]$. Then for all $x \in [a, b]$,*

$$\frac{m}{(n+1)!}(x-a)^{n+1} \leq R_n(x) \leq \frac{M}{(n+1)!}(x-a)^{n+1}. \tag{1}$$

Proof. By induction. For $n = -1$ the result is trivial.

Let $N \geq 0$. Suppose that the theorem holds for all functions f and for $n = N - 1$. Fix a function f for which $m \leq f^{(N+1)}(x) \leq M$ for $x \in [a, b]$. Write $f(x) = P_N(x) + R_N(x)$. Then $f'(x) = P_N'(x) + R_N'(x)$. Note that P_N' is the Taylor polynomial of f' of order $N - 1$, and so R_N' is the corresponding remainder term. By our induction hypothesis (applied to the function f' with $n = N - 1$),

$$\frac{m}{N!}(x-a)^N \leq R_N'(x) \leq \frac{M}{N!}(x-a)^N, \tag{2}$$

for $a \leq x \leq b$. Hence Lemma 2 gives the required inequality. ∎

We conclude with a proof of Lagrange's classical formula. This might be omitted in calculus classes.

Proof. Choose $m = \inf_{a \leq x \leq b}\{f^{(n+1)}(x)\}$ and $M = \sup_{a \leq x \leq b}\{f^{(n+1)}(x)\}$ (if $f^{(n+1)}$ is unbounded, we allow $m, M = \pm\infty$). Thus by Theorem 3, $R_n(b) = \frac{k}{(n+1)!}(b-a)^{n+1}$, where $m \leq k \leq M$. If $m < k < M$, then the assertion follows directly from Darboux's Intermediate Value Theorem. Otherwise, it follows from the following lemma. ∎

Lemma 4. *In Theorem 3 either of the equalities holds (for b) if and only if $f^{(n+1)}$ is constant.*

Proof. It is clear that if $f^{(n+1)}$ is constant, then both equalities hold.

In the converse direction, assume for example that $R_n(b) = \frac{M}{(n+1)!}(b-a)^{n+1}$. Let $Q_n(x) = \frac{M}{(n+1)!}(x-a)^{n+1}$. Then $R_n(a) = Q_n(a) = 0$ and $R_n(b) = Q_n(b)$. Since R_n' is the $(n-1)$-th remainder of f', by Theorem 3

$$R_n'(x) \leq \frac{M}{n!}(x-a)^n = Q_n'(x).$$

Hence $h = Q_n - R_n$ is non-decreasing. Since $h(a) = h(b) = 0$, we get $h(x) = 0$, for all x. This implies that $R_n(x) = Q_n(x)$, so $f = P_n + R_n$ is a polynomial of degree at most $n + 1$. ∎

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