NOTES

Packing Squares in a Square

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If we put two non-overlapping squares (not necessarily the same size) inside a unit square, then the sum of their circumferences is at most 4, the circumference of the unit square. Apparently this problem was first posed around 1932 by Paul Erdős as a problem for high school students in Hungary [**3**]. It was actually the simplest case of a more general Erdős conjecture: if we put $k^2 + 1$ non-overlapping squares inside a unit square, then the total circumference remains at most $4k$ [**4**].

Apparently not much work was done on this conjecture—even the paper by Erdős and Graham [**3**], which starts out by discussing this problem, is mostly about packing *identical unit* squares inside a larger square. In 1995 Erdős, calling it "perhaps undeservedly forgotten" [**2**, as quoted in [**1**]], resurrected the conjecture by offering \$50 for a proof or disproof [**4**]. He and Soifer in [**4**] also considered the more general problem of packing an arbitrary number of squares inside a unit square, not just $k^2 + 1$ squares. They provided lower bounds for the total circumference of the squares, and they conjectured that their lower bounds are actually the best possible.

I first learned of this problem from the paper by Campbell and Staton [**1**], who independently also provided lower bounds for the total circumference. They also conjectured that their lower bounds (identical to those of Erdős and Soifer) are the best possible. Naming a conjecture after four people is a bit unwieldly, so we will use initials and call it the ESCS conjecture. In this note we will *not* prove either the original 1932 Erdős conjecture or the seemingly more general ESCS conjecture, but we will show that they are equivalent. If you can prove one of them, then the the other follows.

The problem

Instead of looking at the circumferences of the squares, we will focus on the lengths of their sides, clearly an unimportant change. Therefore put n squares (not necessarily the same size) inside a unit square, so that these squares share no common interior point. Let $e_1, e_2, \ldots, e_n$ denote the side-lengths of these squares. Define $f(n)$ to be the maximum possible value of $\sum_{i=1}^{n} e_i$. Is there a formula for $f(n)$?

There is a slick proof in [**1**] and [**4**] that $f(k^2) = k$ for all $k \geq 1$: apply the Cauchy-Schwarz inequality to the vectors $(1, 1, \ldots, 1)$ and $(e_1, e_2, \ldots, e_{k^2})$ to get

$$e_1 + e_2 + \cdots + e_{k^2} \leq (1^2 + 1^2 + \cdots + 1^2)^{1/2}(e_1^2 + e_2^2 + \cdots + e_{k^2}^2)^{1/2} \leq k,$$

so $f(k^2) \leq k$. Since the standard $k \times k$ grid reaches this upper bound, we conclude that $f(k^2) = k$.

The original Erdős conjecture is that

$$f(k^2 + 1) = k \quad \text{for all } k \geq 1.$$

The ESCS conjecture can be stated as follows:

$$f(k^2 + 2c + 1) = k + \frac{c}{k} \quad \text{for all } k > |c|.$$

Here c can be any integer, positive or negative (or zero). For example, the conjecture states that $f(k^2 - 1) = k - 1/k$ for all $k > 1$. When $c = 0$, the conjecture states that $f(k^2 + 1) = k$—the original Erdős conjecture. Note that if n is an integer that is not a perfect square, then n lies between two squares of opposite parity, say r^2 and $(r + 1)^2$. Hence either $n - r^2$ is odd or $n - (r + 1)^2$ is odd, so the conjecture provides values of $f(n)$ for all nonsquare integers n. For example, suppose $n = 22$. Now 22 lies between 16 and 25, and in this case it is $22 - 25 = -3$ that is odd. So we put $k = 5$ and $c = -2$ in the formula, and the conjectured value of $f(22)$ is $5 - 2/5 = 4.6$.

By explicit construction, Erdős and Soifer (also Campbell and Staton) showed that $f(k^2 + 2c + 1) \geq k + c/k$ for all $k > |c|$. Thus in order to prove the conjecture, all we need to do is show that $k + c/k$ is an upper bound for $f(k^2 + 2c + 1)$. This is easier said than done. Instead, we will show that if the formula is correct for one particular value of c, then it must be correct for all values of c. In particular, the values conjectured by ESCS follow from the value conjectured originally by Erdős.

An upper bound

We first show how knowing f at one particular value of its argument can be leveraged into an upper bound for f at a different value.

First put n small squares (in some configuration) inside a unit square. Let A denote the sum of the edge-lengths of the n squares, i.e., $A = \sum_{i=1}^{n} e_i$. Set aside this unit square for the moment. Now take another unit square and divide it into the standard $b \times b$ grid of squares, each with side length $1/b$. Remove an $a \times a$ subsquare, and replace it with our first square, shrunk by a factor of b/a so that it fits inside the $a \times a$ space. FIGURE 1 illustrates this for $n = 7$, $b = 5$, and $a = 3$.

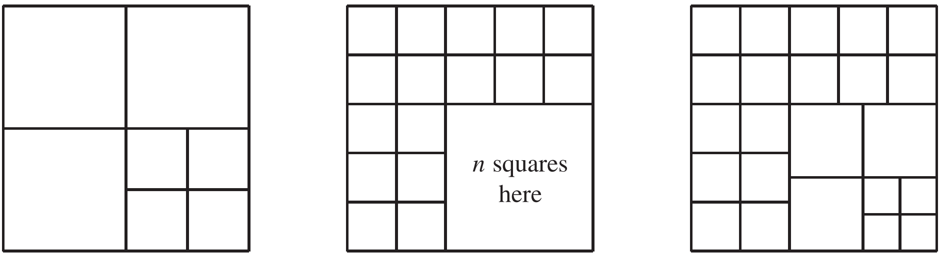

Figure 1 On the left is n squares packed into a unit square. In the center is a $b \times b$ grid with an $a \times a$ subsquare removed. On the right we have filled the $a \times a$ space with a shrunken version of our first square.

We now have a configuration of $b^2 - a^2 + n$ squares inside the unit square. The sum of the side lengths of these squares is $aA/b + (b^2 - a^2)/b$. This is at most $f(b^2 - a^2 + n)$, so we have $aA/b + (b^2 - a^2)/b \leq f(b^2 - a^2 + n)$. Rewriting the inequality gives us

$$A \leq a - \frac{b^2}{a} + \frac{b}{a} f(b^2 - a^2 + n).$$

Since our original packing of n squares in the unit square was arbitrary, we conclude that

$$f(n) \leq a - \frac{b^2}{a} + \frac{b}{a} f(b^2 - a^2 + n). \tag{1}$$

Thus if we know $f(n + b^2 - a^2)$, then we have an upper bound for $f(n)$. Different values of a and b produce different upper bounds; we will make good use of this fact.

The main result and proof

It's probably worthwhile to state our main result in a formal way. For any integer c, write $P(c)$ for the statement

$$f(k^2 + 2c + 1) = k + \frac{c}{k} \quad \text{for all } k > |c|.$$

We will prove that the truth of $P(c)$ for one value of c implies that $P(c)$ is true for all values of c. In particular, if $P(0)$ is true (the original Erdős conjecture), then all of the $P(c)$'s are true (the ESCS conjecture).

Naturally the proof is by induction on c, but in contrast to the usual case, we need to show not only that $P(c-1) \implies P(c)$ (forward induction), but also that $P(c+1) \implies P(c)$ (backward induction). This situation arises because c can be any integer, including negative integers.

We proceed in two steps. In the first step, we derive a crude upper bound for $f(k^2 + 2c + 1)$ based on equation (1) and the induction assumption.

LEMMA 1. *Suppose $P(c-1)$ is true. Then*

$$f(k^2 + 2c + 1) \leq k + \frac{c}{k} + \frac{k+c}{k(k^2-1)} \quad \textit{for all } k > |c|. \tag{2}$$

Similarly, suppose $P(c+1)$ is true. Then

$$f(k^2 + 2c + 1) \leq k + \frac{c}{k} + \frac{k-c}{k(k+1)^2} \quad \textit{for all } k > |c|. \tag{3}$$

Proof. We first assume that $P(c-1)$ is true. Suppose $k > |c|$. Put $n = k^2 + 2c + 1$, $a = k - 1$, $b = k$ in equation (1). Then

$$\begin{aligned} b^2 - a^2 + n &= 2k - 1 + k^2 + 2c + 1 = (k+1)^2 + (2c - 1) \\ &= (k+1)^2 + 2(c-1) + 1. \end{aligned}$$

Note that $k + 1 > |c - 1|$, so we can use our hypothesis that $P(c-1)$ is true, i.e., $f(b^2 - a^2 + n) = k + 1 + (c-1)/(k+1) = k + (k+c)/(k+1)$. Thus equation (1) becomes (after some straightforward algebra)

$$\begin{aligned} f(k^2 + 2c + 1) &\leq k - 1 - \frac{k^2}{k-1} + \frac{k}{k-1}\left(k + \frac{k+c}{k+1}\right) \\ &= k + \frac{c}{k} + \frac{k+c}{k(k^2-1)}, \end{aligned}$$

as claimed. The proof of equation (3) proceeds similarly, but we use $n = k^2 + 2c + 1$, $a = k + 1$, and $b = k + 2$ in equation (1). ■

We thus now have an upper bound for $f(k^2+2c+1)$ that applies for all $k > |c|$, but it is not quite what we want—it is too big by $(k+c)/(k(k^2-1))$. In the second step, we refine the upper bound so that it matches the ESCS lower bound.

LEMMA 2. *Equation* (2) *implies* $P(c)$. *Similarly, equation* (3) *implies* $P(c)$.

Proof. As stated above, it is enough to show that $f(k^2+2c+1) \le k + c/k$.

First assume equation (2) is true. In equation (1) we let $n = k^2 + (2c+1)$ as before, but now let $a = k$ and keep b arbitrary. Then $b^2 - a^2 + n = b^2 + (2c+1)$. Note that we can apply equation (2) to $f(b^2+2c+1)$ since $b > a = k > |c|$. Equation (1) then implies

$$\begin{aligned} f(k^2+2c+1) &\le k - \frac{b^2}{k} + \frac{b}{k} f(b^2+(2c+1)) \\ &\le k - \frac{b^2}{k} + \frac{b}{k}\left(b + \frac{c}{b} + \frac{b+c}{b(b^2-1)}\right) \\ &= k + \frac{c}{k} + \frac{b+c}{k(b^2-1)}. \end{aligned}$$

This is true for any value of $b > k$. Now let $b \to \infty$. We get

$$f(k^2+2c+1) \le k + c/k.$$

which is exactly what we want.

The other half of the lemma is proved similarly. Details are left to the reader. ■

Putting lemmas (1) and (2) together, we get our main theorem.

THEOREM. *If $P(c)$ is true for one value of c, then it is true for all values of c.*

One final note on this topic. Looking carefully at the proof of Lemma (2), we see that in order to prove the ESCS conjecture, it suffices to show that $f(k^2+2c+1) = k + c/k + \epsilon(k)$, where $k\epsilon(k) \to 0$ as $k \to \infty$. Unfortunately, in order to do this it is probably necessary to investigate in detail the placement of the n squares inside the unit square.

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