$$\binom{e_1+e_2+\cdots+e_t}{e_1\,e_2\cdots e_t}=\prod_{i=1}^{t-1}\binom{e_1+e_2+\cdots+e_{i+1}}{e_{i+1}}.$$

By melding this program with one that generates all solutions in nonnegative integers of $e_1+e_2+\cdots+e_t=n$, the reader can verify, as we did, the results of our article for specific values of n, t, and p.

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Volumes of Cones, Paraboloids, and Other "Vertex Solids"

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While performing some calculations involving the volume of a solid circular paraboloid $z(r)=h(1-(r/a)^2)$ (with $h>0$) as illustrations of the Divergence Theorem in vector calculus, the author noticed that the ratio of the volume of the portion of the solid paraboloid above the polar plane to that of the solid cylinder with the same base and height h (its associated solid cylinder) is equal to $1/2$ for all values of the radius a. A natural question is whether this ratio holds for elliptical paraboloids or for paraboloids with any simple curve as a base. Another question is whether there is a similar ratio that is independent of the shape and size of the base for exponents other than 2 in the formula for $z(r)$. Our theorem answers these questions for a class of solids we call vertex solids; a class that includes cones and paraboloids.

We will now define vertex solids. Let $r = g(\theta)$ describe a simple closed curve in the polar plane such that $0 \leqslant g(\theta)$ for $0 \leqslant \theta \leqslant 2\pi$. Let the point V be on the positive z-axis with $z = h$ (this will be the top vertex for the vertex solid). For each fixed θ in $[0, 2\pi]$, consider the curves $z_k(r) = h(1-(r/g(\theta))^k)$ where k is a positive constant (if $g(\theta) = 0$, let $z_k(r) = h$). If k is an integer, then $z_k(r)$ is clearly the unique curve of kth degree that goes through V and the point $(\theta, g(\theta), 0)$ in the polar plane and that has the property $d^i z/dr^i = 0$ at V for $i = 1, 2, \ldots, k-1$ (for fixed θ). Now consider r and θ as independent variables and view the above expression for $z_k = z_k(r, \theta)$ as representing a surface. If $r = g(\theta)$ describes an ellipse, then z_1 represents an elliptical cone and z_2 represents an elliptical paraboloid. For all $k > 0$, we call the solid defined by the set of points (r, θ, z) satisfying $0 \leqslant r \leqslant g(\theta)$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z \leqslant z_k(r, \theta)$ a *vertex solid*. Its *associated solid cylinder* is the set of points (r, θ, z) satisfying $0 \leqslant r \leqslant g(\theta)$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z \leqslant h$.

THEOREM. *The ratio of the volume of the vertex solid of degree k to that of its associated solid cylinder is $k/(k+2)$. (Thus for an elliptical cone the volume is 1/3 that of its associated cylinder's volume πabh; for an elliptical paraboloid its volume is 1/2 of πabh. Here, a and b are the minor and major radii of the ellipse. The ratios 1/3 and 1/2 hold for cones and paraboloids (respectively) with any base that is describable by a simple closed curve.)*

Proof.

$$\text{vol(vertex solid)} = \int_0^{2\pi} \int_0^{g(\theta)} z_k(r, \theta) \cdot r \, dr \, d\theta$$

$$\text{vol(solid cylinder)} = \int_0^{2\pi} \int_0^{g(\theta)} h \cdot r \, dr \, d\theta = A.$$

Substitution of the expression for z_k and a fairly simple integration reveals that the ratio of the volume of the vertex solid to that of its associated solid cylinder is

$$\frac{A - (2/(k+2))A}{A} = \frac{k}{k+2}.$$

Notes:

(1) This result easily can be extended to the case where the base of the vertex solid does not lie below the vertex V. In this case, the vertex solid is not entirely contained in its solid cylinder.

(2) As k increases, the vertex solid occupies more and more of its associated solid cylinder, and in the limit occupies all of it.

(3) Consider the cross sections of the vertex solid and its solid cylinder generated by the plane $\theta = c$ (constant). (Assume $g(c) > 0$.) Denoting them by C_v and C_c, it's easy to see that $\text{area}(C_v)/\text{area}(C_c) = k/(k+1)$.

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