Archimedes' Quadrature of the Parabola: A Mechanical View

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Archimedes (287–212 B.C.) is generally considered the greatest creative genius of the ancient world. In his "Quadrature of the Parabola" (see [**1**] and [**3**]), he found the area of the region bounded by a parabola and a chord. While this is an easy problem for today's student of calculus, its solution in 300 B.C. required considerable mathematical skill. Archimedes' method was to fill the region with infinitely many triangles each of whose area he could calculate, and then to evaluate the infinite sum. In his solution, he stated, without proof, three preliminary propositions about parabolas that were known in his time, but are not widely known today. Two modern presentations of Archimedes' solution have appeared in [**2**] and [**5**], but again, these propositions are stated without proof. It is the purpose of this short paper to prove the ideas presented in these obscure propositions so that a complete presentation of Archimedes' solution can be given. Our proofs are novel in that they are "mechanical"; that is, they use simple ideas from elementary physics rather than geometry. We use the fact that a particle, not acted on by friction, in motion near the surface of the earth, has a parabolic trajectory. The proofs given this way are very simple.

We begin with some terminology and conventions. Throughout, our parabolas all have a vertical axis and open downward. A *parabolic segment* consists of the region bounded by a parabola and the line segment joining two of its points B' and B (see Figure 1). The chord $B'B$ is called the *base* of the parabolic segment, and the point M on the parabolic segment furthest from the base is called the *vertex* of the segment (in general this is not the same as the vertex of the parabola). Observe that the tangent line at M is parallel to $B'B$. (This can be seen by rotating the figure so that the base is horizontal, whence M is a maximum point, and so the tangent line there is also horizontal.) If N denotes the point at which the vertical line through M meets the base $B'B$, then MN is called the *diameter* of the segment. (Notice that the diameter is always parallel to the axis of the parabola.) Also, we call $\Delta B'MB$ the *Archimedean triangle* of the segment. Archimedes' approach was to cover the segment with triangles and get a formula for their combined area.

Motion of a Particle in a Uniform Gravitational Field

We use our knowledge of the motion of a particle in a uniform gravitational field to study the properties of the parabolic curve. In particular, we will use the following fact without further proof:

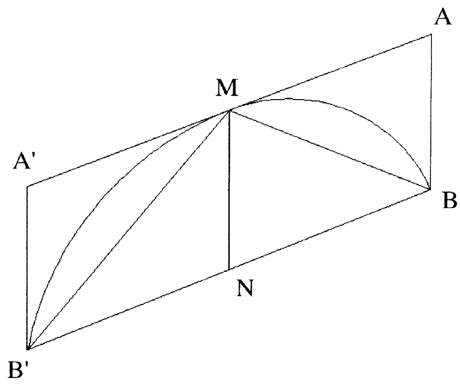

Figure 1. A Parabolic Segment and its Archimedean Triangle

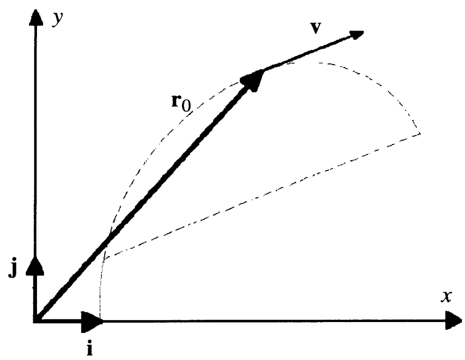

Figure 2. Initial Conditions for Parabolic Flight

Any parabolic curve, in standard position with vertical axis, and opening downward, can be described as the path of a particle starting at the point on the curve $\mathbf{r_0} = x_0\mathbf{i} + y_0\mathbf{j}$ and having initial velocity $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$. (See Figure 2.) The equation of this parabola is given by the parametric equations

$$\begin{cases} x = x_0 + v_x t \\ y = y_0 + v_y t - gt^2/2, \end{cases} \tag{1}$$

where as usual t denotes time and g is the acceleration due to gravity.

If v_x is positive, the particle moves to the right as t increases, and traces the portion of the parabola where $x \geq x_0$. To describe the portion of the parabola to the left of x_0, we can replace $\mathbf{v}$ by $-\mathbf{v}$ and let t increase from zero. (We find it convenient to do this rather than let t be negative.)

We can think of the motion of the particle in vector form in two ways. The first is the usual vector equation

$$\mathbf{r} = (x_0 + v_x t)\mathbf{i} + (y_0 + v_y t - gt^2/2)\mathbf{j}; \tag{2}$$

in the second we view the total motion as the sum of a uniform motion in a straight line and an accelerating vertical drop:

$$\mathbf{r} = (\mathbf{r}_0 + \mathbf{v}t) + \left(\frac{-gt^2}{2}\mathbf{j}\right). \qquad (3)$$

An important feature illustrated by (2) is that the motion is uniform (moves in a straight line with constant velocity) in the x-direction.

Two Preliminary Lemmas

We now develop two lemmas containing the ideas needed for Archimedes' quadrature of the parabola; their content is a combination of the ideas in Archimedes' three original propositions and some of his later derivations. Their proofs use the above ideas from mechanics.

Lemma 1. *If $B'MB$ is a parabolic segment with base $B'B$, vertex M, and diameter MN, then N is the midpoint of $B'B$.*

Proof. Let $\mathbf{r_0}$ be the position vector of the vertex M. Consider a particle launched from M with initial velocity $\mathbf{v}$ in a uniform gravitational field with acceleration $-g$, as in Figure 3. These parameters are selected so that the trajectory of the particle is part of the parabolic segment. Let t_1 denote the time needed for the particle to go from M to B. From equation (3) we see that this motion is the vector sum of the uniform motion $\mathbf{v}t_1$ and the vertical drop $-gt_1^2\mathbf{j}/2$. Notice that the length of NB is $|\mathbf{v}t_1|$.

Similarly, we can generate the left half of the parabolic arc from M to B' by launching a particle from the vertex M with initial velocity $-\mathbf{v}$ (see Figure 4). Since AB and

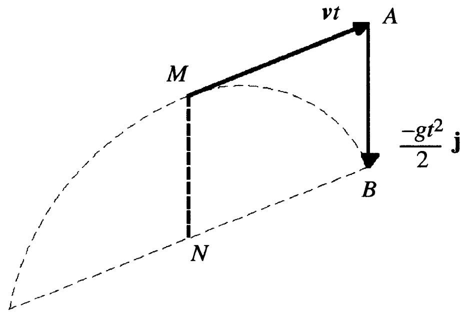

Figure 3. Trajectory from M to B.

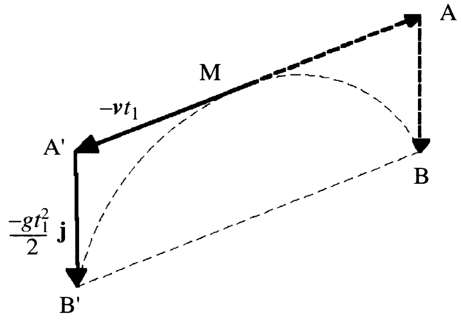

Figure 4. Trajectory from M to B'

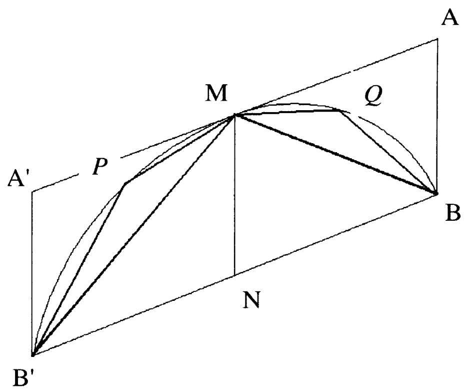

Figure 5. Comparing areas of ΔMQB and $\Delta B'MB$

$A'B'$ have the same length $gt_1^2/2$, the time required to move from M to B' is also t_1. Since $A'M$ and MA have length $|\mathbf{v}t_1|$, it follows that N bisects $B'B$. ■

Lemma 2. *For a parabola in standard position, let M be the vertex of the segment with base $B'B$ and let Q be the vertex of the segment with base MB. Then the area of ΔMQB is one eighth that of $\Delta B'MB$ (see Figure 5).*

Proof. This parabolic arc can be generated as in the previous proof using the trajectory $\mathbf{r} = \mathbf{r_0} + \mathbf{v}t - gt^2\mathbf{j}/2$, and letting t_1 be the time it takes for the particle to move from M to B. The length of AB is $gt_1^2/2$ and the horizontal distance traveled by the particle is $v_x t_1$, where $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$. Thus the area of $\Delta B'MB$ is the same as the area of the parallelogram *NMAB*, which is $(v_x t_1)(gt_1^2/2) = gv_x t_1^3/2$. Since motion is uniform in the horizontal direction, by Lemma 1 the particle travels from M to Q in half the time needed to travel from M to B. Replacing t_1 by the time $t_1/2$ in the formula for the area of $\Delta B'MB$, we see that the area of ΔMQB must be $gv_x(t_1/2)^3/2 = (gv_x t_1^3/2)/8$, which proves the lemma. ■

Let P be the vertex of the parabolic arc $B'PM$ as shown in Figure 5. In the same way the area of triangle $B'PM$ is one eighth the area of triangle $B'MB$.

The Quadrature of the Parabola

We can now find the area of a parabolic segment using a modern version of Archimedes' method of exhaustion.

Archimedes' Theorem. *The area of a parabolic segment is $4/3$ the area of its Archimedean triangle.*

Proof. We exhaust the area of the parabolic segment by the sum of Archimedean triangles. We let α be the area of $\Delta B'MB$. Next we observe that by Lemma 2 the areas of the two triangles $\Delta B'PM$ and ΔMQB are each $\alpha/8$. Adding the areas of these 3 triangles gives us

$$\alpha + \frac{1}{8}\alpha + \frac{1}{8}\alpha = \alpha\left(1 + \frac{1}{4}\right).$$

In the next step we add the areas of the triangles in the parabolic segments with bases $B'P$, PM, MQ and QB. Each of these triangles has area $\frac{1}{8}(\frac{1}{8}\alpha)$, and since there are four of them, their contribution to the total area is $\frac{1}{16}\alpha$. Continuing in this way we get for the total area of the parabolic segment

$$\alpha\left(1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots\right).$$

This is a geometric series whose sum is

$$\frac{1}{1-\frac{1}{4}}=\frac{4}{3},$$

which completes the proof. ■

In our argument, we assumed that the infinity of smaller and smaller triangles fill the parabolic segment, a fact that appears clear from the figure. Archimedes is very careful about this point and gives a proposition due to Eudoxus to explain it.

Final Remarks

In closing we note that Archimedes (see [**1**] and [**3**]) gave a second derivation of the area of a parabolic segment, one that is in fact mechanical. However, it is very different from the analysis given here.

A good collection of elementary mathematical theorems proven by novel mechanical methods by Uspenskii is found in [**4**].

References

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3. T. L. Heath, *The Works of Archimedes with the Method of Archimedes*, Dover Publications, 1912, (reprint of 1897 ed.). pp. 233–252.
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5. R. M. Young, *Excursions in Calculus, An Interplay of the Continuous and the Discrete*, MAA, 1992, pp. 310–314.

Kenneth Yanosko (ky1@humboldt.edu), Professor Emeritus, Humboldt State University, Arcata, CA 95521, sent in this item from *Poseidon's Gold: a Marcus Didius Falco Mystery* by Lindsey Davis, Crown Publishers, 1992, page 159:

"He had always been a failure. The worst kind: someone you could not help but feel sorry for, even while he was messing you about. He was a terrible teacher. He might have been a snappy mathematician, but he could not explain anything. Struggling to make sense of his long-winded diatribes, I had always felt as if he had set me a problem that needed three facts in order to solve it, but he had only remembered to tell me two of them. Definitely a man whose hypotenuse squared had never totaled the squares of his other two sides."

