CLASSROOM CAPSULES

EDITOR

Ricardo Alfaro
University of Michigan–Flint
Flint, MI 48502

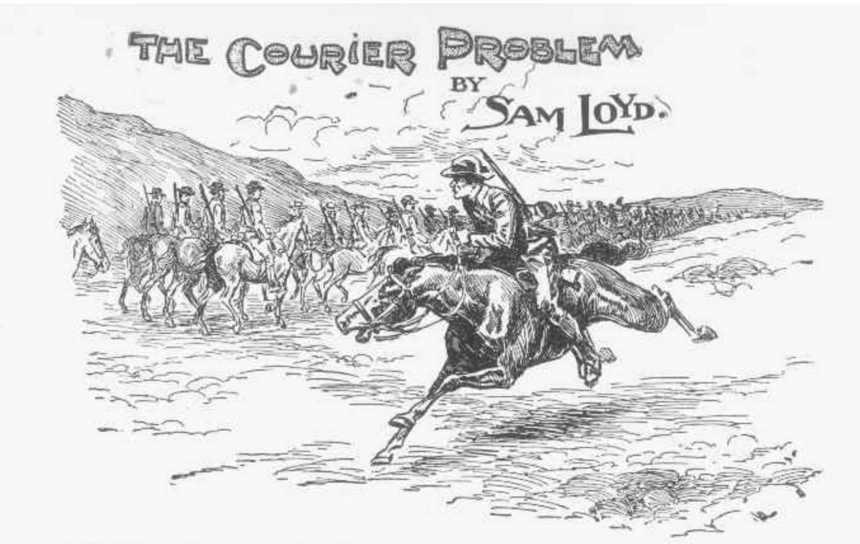
Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor.

Sam Loyd's Courier Problem with Diophantus, Pythagoras, and Martin Gardner

Owen O'Shea (owenoshea4@eircom.net), Cobh, Ireland

In his classic collection *Cyclopedia of Puzzles*, published in 1914, Sam Loyd has two versions of the Courier Problem ([**2**, p. 315]):

> For the reason that many communications are being received relating to a very ancient problem, the authorship of which has been incorrectly accredited to me, occasion is taken to present the original version which has led to considerable discussion. It has been reproduced, in many forms, generally accompanied by an absurd statement regarding the impossibility of solving it, which produced letters of inquiry as well as correct answers from some, who, under the misapprehension of having mastered a hitherto unsolved problem, desire to have the same published.



PROPOSITION—An army 50 miles long advances 50 miles while a courier goes around it.

It is a simple and pretty problem which yields readily to ordinary methods, and can be solved by experimental analysis upon the plan generally adopted by puzzlists. The trouble is that the terms of the problem are seldom given correctly and are not generally understood, for which reason, . . . , we will first look at the ancient version which appears in the oldest mathematical works:

A courier starting from the rear of a moving army, fifty miles long, dashes forward and delivers a dispatch to the front and returns to his position in the rear, during the exact time it required the entire army to advance just fifty miles.

How far did the courier have to travel in delivering the dispatch, and returning to his previous position in the rear of the army?

A better puzzle is created by the following extension of the theme given as problem No. 2:

If a square army, fifty miles long by fifty miles wide, advances fifty miles while a courier makes the complete circuit of the army and returns to the starting point in the rear, how far does the courier have to travel?

The problem appeared in print long before Loyd published it. The first known appearance was in *A Companion to the Gentleman's Diary* of 1798, and a year later it was published in *The Gentleman's Mathematics Companion*. It also appeared much later in William R. Ransom's *One Hundred Mathematical Curiosities* in 1955 [**3**]. Loyd describes the puzzle as an ancient problem and gives the answers to both parts. However, he does not indicate how to tackle the problems.

In his book *More Mathematical Puzzles of Sam Loyd* [**1**, p. 103], Martin Gardner discusses both problems and their solutions. The answer to the first question is that the courier travels $1+\sqrt{2}$ times the length of the army, or about 120.71 miles for a fifty-mile-long army. Here is Gardner's solution: For convenience, assume that the length of the army is 1 unit, and also assume that the time it takes the army to march its length is 1. Thus its speed is also 1. Let x denote the total distance traveled by the courier, so that is also his speed. On his forward trip, the courier's speed relative to the army is thus $x-1$, and on the return trip it is $x+1$. Since each trip is a distance of 1 relative to the moving army and the courier's total journey is completed in unit time, it follows that

$$\frac{1}{x-1}+\frac{1}{x+1}=1,$$

or equivalently, $x^2-2x-1=0$. Consequently, $x=1+\sqrt{2}$. This means that in Loyd's problem the courier travels $50\big(1+\sqrt{2}\big)$ miles, or, as asserted earlier, about 120.71 miles.

Gardner's solution to the second question is similar. As before, let the length of the army and the time it takes to travel its length be 1 unit, so its speed is also 1. Additionally, we let x be the total distance traveled by the courier and also his speed. On the forward trip, the courier's speed relative to the army is $x-1$, on the return trip, $x+1$, and on each of the diagonal portions, $\sqrt{x^2-1}$. Since each trip has length 1 relative to the army and the total journey is completed in unit time, it follows that

$$\frac{1}{x-1}+\frac{1}{x+1}+\frac{2}{\sqrt{x^2-1}}=1.$$

This time a fourth-degree equation results: $x^4-4x^3-2x^2+4x+5=0$. The only solution that fits the conditions of the problem is approximately 4.18. Thus, in traversing the entire army, the courier now travels approximately 4.18 times as far as the army, or about 209 miles.

In the spirit of classroom exercises, it would be nice to have versions of the courier problems with integer solutions; that is, Diophantine problems. In exploring this possibility, we discovered that certain Pythagorean triples, such as $(3, 4, 5)$, $(5, 12, 13)$, and $(7, 24, 25)$, that is, those in which the two largest numbers differ by 1, can be used to generate such problems.

Army in single file. Assume that an army 60 miles long is marching at a constant rate. A courier, also going at a constant rate, rides from the rear of the army up to the front, delivers a message, and returns, arriving at the rear just as the army has gone 45 miles. How far did the courier go?

Following Gardner, we again let the speed of the army (but not its length) be 1, and we let x be the speed of the courier. As before, the speed of the courier relative to the army is $x-1$ on the outward journey and $x+1$ on the return journey, so the total time that he travels is $\frac{60}{x-1}+\frac{60}{x+1}$. This must equal 45, the time it takes the army to go the given distance. Gardner's equation is therefore

$$\frac{60}{x-1}+\frac{60}{x+1}=45.$$

This reduces to $3x^2-8x-3=0$, for which the only realistic solution is $x=3$. Thus, the courier travels three times as fast as the army, and therefore rides 135 miles. We thus have a Diophantine version of the courier problem.

Consider now a general single-file problem, in which the army is l miles long and travels d miles during the time that the courier makes the trip. Gardner's equation becomes

$$\frac{l}{x-1}+\frac{l}{x+1}=d.$$

Setting $\alpha=\frac{l}{d}$, we find that this reduces to

$$x^2-2\alpha x-1=0,$$

which has $x=\alpha+\sqrt{\alpha^2+1}$ as its only positive solution.

This is where Pythagorean triples come to the fore. If $(a, b, b+1)$ is such a triple and $\alpha=\frac{b}{a}$, then $x=a$ is a solution to Gardner's equation. Our example used the triple $(3, 4, 5)$ with distances multiplied by 15. Since in these triples $b=\frac{1}{2}(a^2-1)$ and $c=\frac{1}{2}(a^2+1)$, we can pose a single-file courier problem with an integer solution for any odd positive integer a and any constant k:

Problem 1 (single file army). An army $\frac{1}{2}(a^2-1)k$ miles long travels a distance of ak miles during the time that the courier makes his journey. How far does the courier travel?

Army in square formation. We can do something similar for the second of Loyd's problems. Here is an example (also based on the $(3, 4, 5)$ triple):

An army in the formation of a square 36 miles on a side is advancing at a constant rate. A courier, also moving at a steady rate, travels all the way around the army just as it advances 256 miles. How far does the courier go?

Again we take the speed of the army to be 1 and that of the courier x, so Gardner's equation becomes

$$\frac{36}{x-1}+\frac{36}{x+1}+\frac{2\cdot 36}{\sqrt{x^2-1}}=256.$$

In polynomial form, this is

$$81x^4-576x^3-162x^3+576x+1105=0,$$

but all we care about is that $x=\frac{5}{4}$ is a root (this is easy to check in the original equation, not so easy to discover).

The same Pythagorean triples (a,b,c) as before (with a an odd integer, $b=\frac{1}{2}(a^2-1)$, and $c=\frac{1}{2}(a^2+1)$) all generate square-army courier problems with Diophantine solutions:

Problem 2 (square army). An army b^2 miles on a side advances $2a\,(b+c)$ miles during the time that a courier completes his circuit. How far does the courier go?

Gardner's equation for this army can be written as

$$2b^2\left(\frac{x}{x^2-1}+\frac{1}{\sqrt{x^2-1}}\right)=2a\,(b+c)\,,$$

which has $x=\frac{c}{a}$ as a solution. That is, the courier travels $\frac{c}{a}$ times as fast as the army and hence goes $2c(b+c)$ miles.

Note that the roles of a and b can be interchanged here; that is, the army could be b^2 on a side and travel $2b(a+c)$ miles while the courier completes his circuit.

Army in rectangular formation. Of course, armies do not always march in either single-file or square formation; sometimes they are proper rectangles. Suppose that an army is l miles long and w miles wide and that it goes a distance d in the time that the courier transverses it, going at a speed x times that of the army. Gardner's equation for this situation is

$$\frac{l}{x-1}+\frac{l}{x+1}+\frac{2w}{\sqrt{x^2-1}}=d.$$

For example, assume that the army is 40 miles long and 30 miles wide, and that it travels 120 miles while the courier completes his circuit. Then the courier rides $5/3$ as fast as the army travels, and hence goes 200 miles.

Thus, Pythagorean triples of the given type can be used in this equation to generate Diophantine solutions for a rectangular army in two ways, with (a,b,c) again a Pythagorean triple with $c-b=1$.

Problem 3 (rectangular army). An army's length is a multiple of $\frac{1}{2}b^2$ and its width a multiple of $\frac{1}{2}b$. It advances $2a\,(b+c)$ miles during the time that courier completes his circuit. How far does the courier travel?

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References

1. Martin Gardner, *More Mathematical Puzzles of Sam Loyd*, Dover, 2007.
2. Sam Loyd, *Cyclopedia of Puzzles*, Lamb Publishing, 1914. http://www.mathpuzzle.com/loyd/
3. William R. Ransom, *One Hundred Mathematical Curiosities*, J. Weston Walch, 1955.

Beyond the Basel Problem: Sums of Reciprocals of Figurate Numbers

Lawrence Downey (lmd108@psu.edu) and Boon W. Ong (bwo1@psu.edu), Penn State Erie, Behrend College, Erie, PA 16563, and James A. Sellers (sellersj@math.psu.edu), Penn State University, University Park, PA 16802

Throughout 2007, a great deal of attention was paid to the life and work of Leonhard Euler (1707–1783), and rightly so! Euler's enormous impact can certainly still be felt today. And while his work spans a great many areas of interest within mathematics, here we focus on one of his earliest pursuits—determining the sums of particular infinite series.

Summing infinite series was a hot topic in the late 17th and early 18th centuries. Indeed, Jacob Bernoulli's *Tractatus de Seriebus Infinitus* [**1**] was of momumental importance in the field. Bernoulli determined the sums of numerous convergent series very elegantly, making particularly good use of what we now call "telescoping". For example, he proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1,$$

a fact often demonstrated to calculus students. Of course, this means that the sum of the reciprocals of the triangular numbers, the numbers 1, 3, 6, 10, 15, . . . , is 2.

Another series of interest at the time was

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

and finding its sum became known as the Basel Problem. Its solution eluded the best mathematical minds of the day. (See, for example, Dunham [**4**, Ch. 3] for more information.) It was accepted that the series converges, and many had approximated its sum with decent accuracy, but no one was able to find its exact value. Enter Leonhard Euler.

In 1737, Euler provided the first of his many proofs of the problem. In this early work, which put him on the map in the eyes of the mathematical community, Euler proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

a result often quoted, but (sadly) rarely proved, in calculus courses.