A Method for Vector Proofs in Geometry

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In his book on problem-solving, Loren C. Larson observes that, for proofs in plane geometry, it is often convenient to have a notation to represent the rotation of a vector through 90° (see [**6**], Example 8.3.5, pp. 305–6). Rather than introduce such a notation and face the necessity of developing both its properties and facility in its use, we suggest the following natural alternative which relies on the exploitation of standard properties of the vector cross-product, and which can lead to elegant vector proofs. Our approach also serves as an alternative to introducing complex numbers, where multiplication by i corresponds to rotation through 90° (as utilised, for example, in one of the standard proofs of Napoleon's Theorem—see [**6**], Example 8.4 pp. 314–5 and compare with our proof below), and can achieve a succinctness close to that attained by using two-dimensional transformation geometry in the spirit of [**2**], [**3**], and [**5**].

We start by reworking Larson's Example 8.3.5 (a problem originally due to M. Slater in [**10**]) using our method, and then give several further examples to illustrate its message: that it may sometimes be helpful to 'three-dimensionalise' the proofs of certain results in plane geometry by introducing a fixed vector perpendicular to the plane and that such a vector, far from complicating the solution, may actually serve as a catalyst in the proof.

In Slater's Problem, we are asked to prove: *if similar isosceles triangles OAB', OBA', ABO' are erected on the sides of a triangle OAB (respectively externally, externally, internally), as in* FIGURE 1, *then $OA'O'B'$ is a parallelogram.*

Introduce $\mathbf{k}$, a unit vector 'up' out of the plane of $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. Then $\mathbf{k}\times\mathbf{a}, \mathbf{k}\times\mathbf{b}$ are normal to $\mathbf{a}, \mathbf{b}$, respectively, and lie in the plane of $\mathbf{a}$ and $\mathbf{b}$.

Similarity of the constructed isosceles triangles implies the existence of a positive scalar m such that:

$$\overrightarrow{OB'} = \tfrac{1}{2}\mathbf{a} + m(\mathbf{k}\times\mathbf{a})$$

$$\overrightarrow{OA'} = \tfrac{1}{2}\mathbf{b} - m(\mathbf{k}\times\mathbf{b})$$

$$\overrightarrow{OO'} = \tfrac{1}{2}(\mathbf{a}+\mathbf{b}) + m\mathbf{k}\times(\mathbf{a}-\mathbf{b}).$$

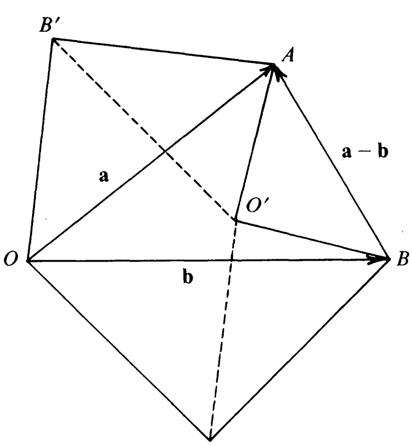

FIGURE 1

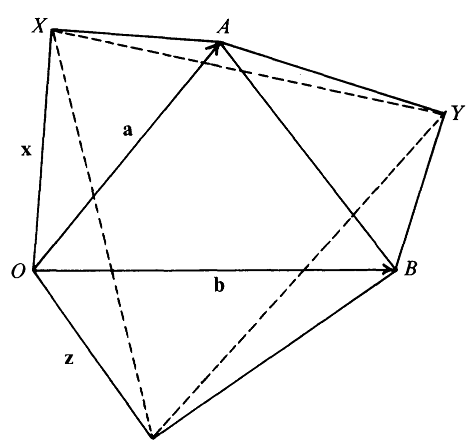

FIGURE 2

It is thus clear that $\overrightarrow{OO'} = \overrightarrow{OA'} + \overrightarrow{OB'}$ and hence that $OA'O'B'$ is a parallelogram.

Notice how smoothly a scaling factor can be incorporated into **k** and the cross product terms; this will be a recurring feature of the proofs that follow, too.

The same approach works as well on Exercises 8.3.12–8.3.16 in Larson's book and, as a second example, we prove an extension of Exercise 8.3.15 attributed to W. L. Ferrar by E. A. Maxwell in [**8**]. *Given any triangle OAB, let X, Y, Z denote the new vertices of externally erected similar triangles with bases OA, AB, OB.* (See FIGURE 2.) *Then triangles XYZ and OAB have the same centroid.*

Here, we introduce the unit vector **k** as before and observe that similarity of the constructed (not necessarily isosceles!) triangles this time implies the existence of two positive scalars m, n such that:

$$\mathbf{x} = \overrightarrow{OX} = m\mathbf{a} + n(\mathbf{k} \times \mathbf{a})$$
$$\mathbf{y} = \overrightarrow{OY} = \mathbf{a} + m(\mathbf{b} - \mathbf{a}) + n\mathbf{k} \times (\mathbf{b} - \mathbf{a})$$
$$\mathbf{z} = \overrightarrow{OZ} = (1 - m)\mathbf{b} - n\mathbf{k} \times \mathbf{b}.$$

Adding these equations gives $\frac{1}{3}(\mathbf{x} + \mathbf{y} + \mathbf{z}) = \frac{1}{3}(\mathbf{a} + \mathbf{b})$ or, equivalently, by a standard vector characterization of the centroid, that XYZ and OAB have the same centroid.

Our remaining examples all involve use of the triple scalar product. We write $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ to denote the triple scalar product, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, of three vectors and to emphasize two properties that we shall use repeatedly in what follows:

(i) The triple scalar product is linear in each term (meaning, for example, that $[\mathbf{a} + m\mathbf{a}', \mathbf{b}, \mathbf{c}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}] + m[\mathbf{a}', \mathbf{b}, \mathbf{c}]$), and

(ii) The triple scalar product changes sign if two vectors are interchanged (in particular, it vanishes if any two constituent vectors are equal).

We also employ, without further comment, the trivial observation that

(iii) $(\mathbf{k} \times \mathbf{a}) \cdot (\mathbf{k} \times \mathbf{b}) = |\mathbf{k}|^2(\mathbf{a} \cdot \mathbf{b})$ if **k** is perpendicular to **a** and **b**.

With these remarks in mind, we give a proof of the following result from [**2**]: *if, on adjacent sides AC and CB of the parallelogram $ACBO$, external equilateral triangles ACX and CBY are erected, then triangle OXY is also equilateral.* (See FIGURE 3.)

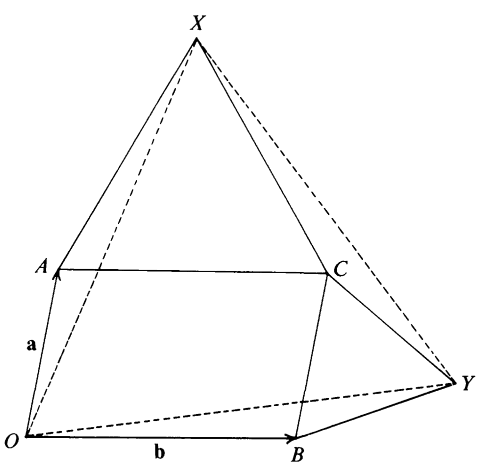

FIGURE 3

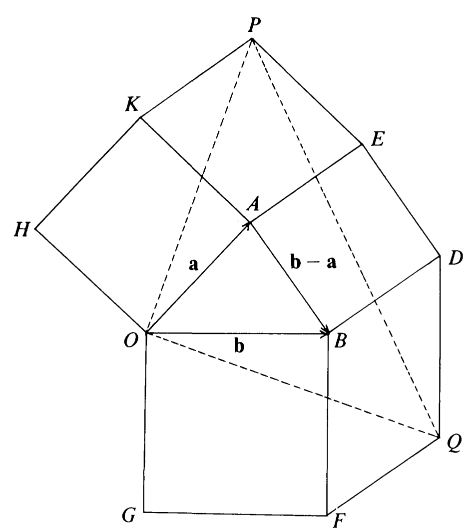

FIGURE 4

This time, take $\mathbf{k}$ to be 'up' out of the plane of $\mathbf{a}$ and $\mathbf{b}$ and of magnitude $\frac{1}{2}\sqrt{3}$. The position vectors of X and Y are then easily seen to be:

$$\mathbf{x} = \mathbf{a} + \tfrac{1}{2}\mathbf{b} + (\mathbf{k} \times \mathbf{b})$$
$$\mathbf{y} = \tfrac{1}{2}\mathbf{a} + \mathbf{b} - (\mathbf{k} \times \mathbf{a}),$$

whence

$$\mathbf{x} - \mathbf{y} = \tfrac{1}{2}\mathbf{a} - \tfrac{1}{2}\mathbf{b} + \mathbf{k} \times (\mathbf{a} + \mathbf{b}).$$

Thus

$$\begin{aligned} |\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x} &= \left(\mathbf{a} + \tfrac{1}{2}\mathbf{b} + \mathbf{k} \times \mathbf{b}\right) \cdot \left(\mathbf{a} + \tfrac{1}{2}\mathbf{b} + \mathbf{k} \times \mathbf{b}\right) \\ &= |\mathbf{a}|^2 + \tfrac{1}{4}|\mathbf{b}|^2 + |\mathbf{k} \times \mathbf{b}|^2 + \mathbf{a} \cdot \mathbf{b} + 2[\mathbf{a},\mathbf{k},\mathbf{b}] \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 + \mathbf{a} \cdot \mathbf{b} + 2[\mathbf{a},\mathbf{k},\mathbf{b}]. \end{aligned}$$

But the latter expression is unchanged by the substitutions

$$\begin{cases} \mathbf{a} \to \ \ \mathbf{a} + \mathbf{b} \\ \mathbf{b} \to -\mathbf{a} \end{cases} \text{and} \quad \begin{cases} \mathbf{a} \to -\mathbf{b} \\ \mathbf{b} \to \ \ \mathbf{a} + \mathbf{b} \end{cases}$$

corresponding to the evaluation of $|\mathbf{y}|^2$ and $|\mathbf{x} - \mathbf{y}|^2$, respectively, which establishes the result.

In a similar vein, we prove the somewhat surprising theorem stated in Ogilvy ([**9**], p. 120). *Given any triangle OAB, construct the exterior squares $ABDE$, $OBFG$, $AOHK$ and complete the parallelograms $FBDQ$, $EAKP$.* (See FIGURE 4.) *Then POQ is always a right-angled isosceles triangle.*

In this case, let $\mathbf{k}$ be a unit vector 'up' out of the plane of $\mathbf{a}$ and $\mathbf{b}$. The position vectors $\mathbf{p}, \mathbf{q}$ of P and Q are then given by:

$$\mathbf{p} = \mathbf{a} + (\mathbf{k} \times \mathbf{a}) + \mathbf{k} \times (\mathbf{b} - \mathbf{a}) = \mathbf{a} + \mathbf{k} \times \mathbf{b},$$
$$\mathbf{q} = \mathbf{b} + (\mathbf{b} \times \mathbf{k}) + \mathbf{k} \times (\mathbf{b} - \mathbf{a}) = \mathbf{b} - \mathbf{k} \times \mathbf{a}.$$

Then, as above, $|\mathbf{p}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2[\mathbf{a},\mathbf{k},\mathbf{b}]$, which, being invariant under the substitution

$$\begin{cases} \mathbf{a} \to \ \ \mathbf{b} \\ \mathbf{b} \to -\mathbf{a}, \end{cases}$$

corresponding to the evaluation of $|\mathbf{q}|^2$, shows that $|\mathbf{p}| = |\mathbf{q}|$. Also, $\mathbf{p} \cdot \mathbf{q} = \mathbf{b} \cdot \mathbf{a} - (\mathbf{k} \times \mathbf{b}) \cdot (\mathbf{k} \times \mathbf{a}) = \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{a} = 0$, which concludes the proof. (Here, the vector proof is noticeably shorter and more elegant than a 'Euclidean' proof: compare [**9**], p. 169.)

We next fulfill our promise in the introduction by providing a proof of 'Napoleon's Theorem': *the centroids of equilateral triangles erected (externally) on the sides of any triangle again form an equilateral triangle.*

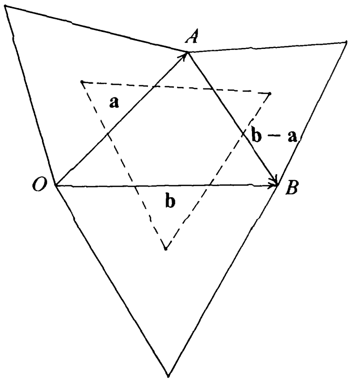

FIGURE 5

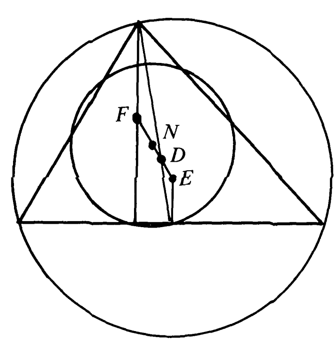

FIGURE 6

With notation as in FIGURE 5, let $\mathbf{k}$ be a vector 'up' out of the plane of $\mathbf{a}$ and $\mathbf{b}$, of magnitude $\sqrt{3}/6$. Then the position vectors of the three centroids involved are readily seen to be $\frac{1}{2}\mathbf{a} + (\mathbf{k} \times \mathbf{a})$, $\frac{1}{2}\mathbf{b} - (\mathbf{k} \times \mathbf{b})$, $\frac{1}{2}(\mathbf{a}+\mathbf{b}) + \mathbf{k} \times (\mathbf{b}-\mathbf{a})$, so that the sides of the triangle formed by these centroids have representing vectors $\frac{1}{2}\mathbf{a} + \mathbf{k} \times (2\mathbf{b} - \mathbf{a})$, $\frac{1}{2}\mathbf{b} + \mathbf{k} \times (\mathbf{b} - 2\mathbf{a})$ and $\frac{1}{2}(\mathbf{a}-\mathbf{b}) + \mathbf{k} \times (\mathbf{a}+\mathbf{b})$. A calculation along now familiar lines quickly shows that:

$$\begin{aligned}3\left|\tfrac{1}{2}\mathbf{a} + \mathbf{k} \times (2\mathbf{b}-\mathbf{a})\right|^2 &= 3\left(\tfrac{1}{4}|\mathbf{a}|^2 + \tfrac{1}{12}(2\mathbf{b}-\mathbf{a})\cdot(2\mathbf{b}-\mathbf{a}) + 2[\mathbf{a},\mathbf{k},\mathbf{b}]\right)\\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b} + 6[\mathbf{a},\mathbf{k},\mathbf{b}]\end{aligned}$$

which, being invariant under the respective substitutions

$$\begin{cases}\mathbf{a} \to \mathbf{b}\\ \mathbf{b} \to \mathbf{b} - \mathbf{a}\end{cases} \quad \text{and} \quad \begin{cases}\mathbf{a} \to \mathbf{a} - \mathbf{b}\\ \mathbf{b} \to \mathbf{a},\end{cases}$$

establishes that the sides of the triangle concerned have equal length. (The proof is readily modified to obtain the analogous theorem for internally erected equilateral triangles.)

For a more classical application, we next give a short proof of the ratio properties of the Euler Line. We take for granted the existence of the centroid D, circumcentre E, orthocentre F and nine-point centre N of a triangle (recall that the latter may be defined as the centre of that circle which, rather impressively, can always be drawn through the midpoints of sides and feet of altitudes of a triangle—see [**9**], pp. 117–20 for further details), and aim to prove the theorem that *D, E, F, N lie on a line (the Euler Line of the triangle) with* $ED:DN:NF = 2:1:3$. (See FIGURE 6.)

We label the vertices of the triangle O, A, B and use the obvious letters to denote the position vectors of points relative to O. Take $\mathbf{k}$ to be a fixed (non-zero) vector normal to the plane of $\mathbf{a}$ and $\mathbf{b}$.

The position vector of the centroid of OAB is given by $\mathbf{d} = \frac{1}{3}(\mathbf{a}+\mathbf{b})$. The position of the circumcentre is determined by the point of intersection of the perpendicular bisectors of OA and OB, that is, from the existence of scalars p, q such that $\mathbf{e} = \frac{1}{2}\mathbf{b} + p(\mathbf{b} \times \mathbf{k}) = \frac{1}{2}\mathbf{a} + q(\mathbf{a} \times \mathbf{k})$; from which $\frac{1}{2}\mathbf{b}\cdot\mathbf{b} + p[\mathbf{b},\mathbf{k},\mathbf{b}] = \frac{1}{2}\mathbf{a}\cdot\mathbf{b} + q[\mathbf{a},\mathbf{k},\mathbf{b}]$ and hence

$$q = \frac{(\mathbf{b}-\mathbf{a})\cdot\mathbf{b}}{2[\mathbf{a},\mathbf{k},\mathbf{b}]}.$$

The position of the orthocentre is determined by the point of intersection of the altitudes through A and B, that is, from the existence of scalars r, s such that $\mathbf{f} = \mathbf{a} + r(\mathbf{b} \times \mathbf{k}) = \mathbf{b} + s(\mathbf{a} \times \mathbf{k})$, from which

$$s = \frac{(\mathbf{a}-\mathbf{b})\cdot\mathbf{b}}{[\mathbf{a},\mathbf{k},\mathbf{b}]} = -2q.$$

Similarly, using the definition of the nine-point centre given above, we obtain $\mathbf{n} = \frac{1}{4}\mathbf{a} + \frac{1}{2}\mathbf{b} - \frac{1}{2}q(\mathbf{a} \times \mathbf{k})$. Thus:

$$\begin{aligned}\mathbf{d} &= \tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b}\\ \mathbf{e} &= \tfrac{1}{2}\mathbf{a} \qquad\quad + q(\mathbf{a} \times \mathbf{k})\\ \mathbf{f} &= \qquad\quad \mathbf{b} - 2q(\mathbf{a} \times \mathbf{k})\\ \mathbf{n} &= \tfrac{1}{4}\mathbf{a} + \tfrac{1}{2}\mathbf{b} - \tfrac{1}{2}q(\mathbf{a} \times \mathbf{k})\end{aligned}$$

and the required property:

$$\tfrac{1}{3}(\mathbf{f}-\mathbf{n}) = \mathbf{n} - \mathbf{d} = \tfrac{1}{2}(\mathbf{d}-\mathbf{e})$$

follows immediately.

It is interesting to compare this proof with the more standard vector approaches to the theorem in [**7**] and [**11**].

As our penultimate example we tackle the problem in [**1**]: *if, in the isosceles triangle OAB of* FIGURE 7 *with circumcentre $E, OA = OB$, D is the midpoint of OA and C is the centroid of triangle OBD, then EC and BD are perpendicular.*

Retaining the notation of the previous example, we have:

$$\mathbf{e}=\frac{1}{2}\mathbf{a}+\frac{(\mathbf{b}-\mathbf{a})\cdot\mathbf{b}}{2[\mathbf{a},\mathbf{k},\mathbf{b}]}(\mathbf{a}\times\mathbf{k}) \qquad \text{and} \qquad \mathbf{c}=\frac{1}{3}\left(\mathbf{b}+\frac{1}{2}\mathbf{a}\right),$$

whence

$$\overrightarrow{EC}=\frac{1}{3}\mathbf{a}-\frac{1}{3}\mathbf{b}+\frac{(\mathbf{b}-\mathbf{a})\cdot\mathbf{b}}{2[\mathbf{a},\mathbf{k},\mathbf{b}]}(\mathbf{a}\times\mathbf{k}).$$

Since $\overrightarrow{BD}=\frac{1}{2}\mathbf{a}-\mathbf{b}$ we deduce:

$$\begin{aligned}6\overrightarrow{EC}\cdot\overrightarrow{BD} &= 3\overrightarrow{EC}\cdot 2\overrightarrow{BD}\\ &= |\mathbf{a}|^2-\mathbf{a}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b}+2|\mathbf{b}|^2-3(\mathbf{b}-\mathbf{a})\cdot\mathbf{b}\\ &= |\mathbf{a}|^2-|\mathbf{b}|^2\\ &= 0\end{aligned}$$

because, and indeed only because, $|\mathbf{a}|=|\mathbf{b}|$.

Finally, we rework problem 1966/6 in [**4**], p. 8 (see also pp. 94–7): *if, in the interior of sides OA, AB, BO of triangle OAB points P, Q, R respectively are selected, then the area of at least one of the triangles BRQ, OPR, APQ is less than or equal to one-quarter of the area of OAB.* (See FIGURE 8.)

As usual, we take $\mathbf{k}$ to be a unit vector 'up' out of the plane of $\mathbf{a}$ and $\mathbf{b}$. By hypothesis, there are scalars x, y, z with $0 < x, y, z < 1$, and $\mathbf{p}=x\mathbf{a}$, $\mathbf{r}=y\mathbf{b}$, $\mathbf{q}=\mathbf{a}+z(\mathbf{b}-\mathbf{a})$.

Then, writing ΔOAB to denote the area of triangle OAB, it is easily checked that:

$$\begin{aligned}&\Delta OAB=\tfrac{1}{2}[\mathbf{b},\mathbf{a},\mathbf{k}]=c,\text{ say},\\ &\Delta OPR=\tfrac{1}{2}xy[\mathbf{b},\mathbf{a},\mathbf{k}]=xyc,\\ &\Delta APQ=\tfrac{1}{2}[-(1-x)\mathbf{a},z(\mathbf{b}-\mathbf{a}),\mathbf{k}]=\tfrac{1}{2}(1-x)z[\mathbf{b},\mathbf{a},\mathbf{k}]=(1-x)zc,\\ &\Delta BRQ=\tfrac{1}{2}[(1-z)(\mathbf{a}-\mathbf{b}),-(1-y)\mathbf{b},\mathbf{k}]=\tfrac{1}{2}(1-z)(1-y)[\mathbf{b},\mathbf{a},\mathbf{k}]=(1-z)(1-y)c.\end{aligned}$$

But the product of the coefficients of c in the last three equations is

$$xy(1-x)z(1-z)(1-y)=x(1-x)y(1-y)z(1-z),$$

and this product must be less than or equal to $\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}$, by the inequality of arithmetic and geometric means. Thus at least one of xy, $(1-x)z$, $(1-z)(1-y)$ is less than or equal to a quarter, as required.

Further geometric theorems which are amenable to proof by our methods may be found in [**6**] (pp. 310–11) and [**5**].

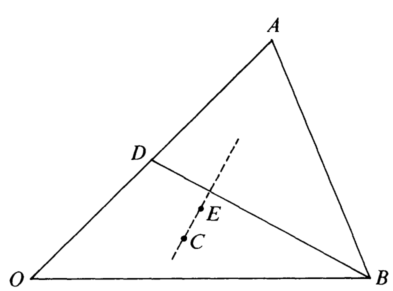

FIGURE 7

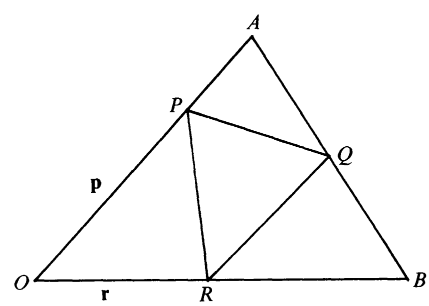

FIGURE 8

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Incenters and Excenters Viewed from the Euler Line

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In the geometry of the triangle the relative positions of the circumcenter O, the centroid G, the nine-point center N, and the orthocenter H have been known for over a century and a half. They form the Euler line and they are so spaced that $OG:GN:NH=2:1:3$. Euler also found other formulas for the distances between these points and for their distances from the incenter I_0 and from the excenters I_1, I_2, I_3. For a quick refresher course on terminology, readers should see **[5]**. Only recently have the full implications of such results on the possible positions of these so-called 'tritangent centers' relative to the Euler line been clarified **[2]**, **[3]**. This clarification came through the investigation of a problem posed by Wernick **[6]** in this MAGAZINE—to reconstruct a triangle given its incenter and any two of the centers forming the Euler line. Since the Euler line of an equilateral triangle is indeterminate (the points O, G, N, H coincide) we exclude the equilateral case from consideration.

There are two main results presented in [3]. First, that *the incenter I_0 always lies inside the circle on diameter GH, and the excenters I_1, I_2, I_3 all lie outside it*. Second, that *relative to the Euler line there is a curiously shaped region, bounded by a closed quartic curve, inside which no tritangent center can lie*. Once it is envisaged, the first result is not difficult to prove, but the original proof of the second result involved truly horrendous algebraic calculations in finding and factorizing the discriminant of a cubic for the cosines of the angles of the triangle.

The object of this note is to give an easier and more geometric (or trigonometric) approach to the second result, together with a simple ruler-and-compass method of constructing points on the boundary of the forbidden zone for tritangent centers.

Let the triangle be ABC, and R the circumradius, r the inradius, r_1 the exradius corresponding to the excenter I_1, and let K and L be the points on the Euler line which are respectively the reflections of H and G in O; that is to say $KL:LO:OG:GN:NH=4:2:2:1:3$. We use Σ to denote cyclic sums over the angles A, B, C of the triangle, i.e., $\Sigma\cos A$ means $\cos A+\cos B+\cos C$. Then standard trigonometric results are **[4]**: