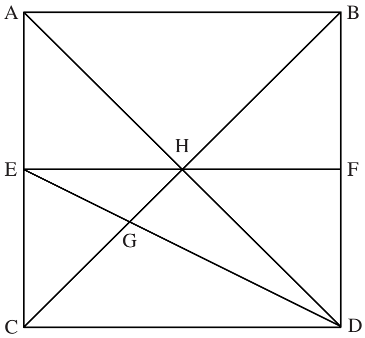
be one-third the length of CH. As CH = DH, GH is also one-third the length of DH. Note that angle DHG is a right angle and also that vertical angles DGH and CGE are equal in measure. Hence, from those facts and right triangle DHG, the tangent of angle CGE = the tangent of angle DGH = DH/GH = 3. ■



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A Bug Problem

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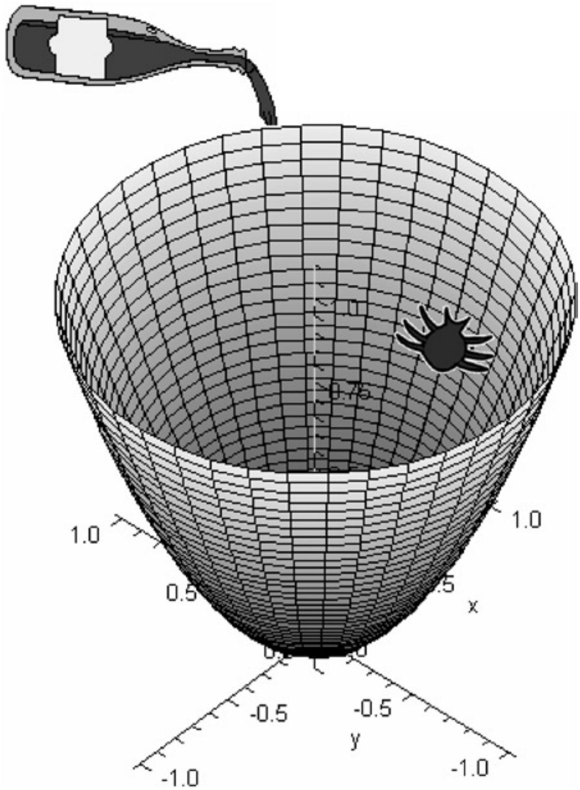
Imagine a vessel, obtained by revolving the function $y = x^2$ around the y-axis. On the wall on the inside of this vessel sits a bug, which becomes fairly unhappy when a liquid of your choice is poured into the vessel at a constant rate of ρ liters/second. Naturally, the bug will crawl upward to avoid getting its feet wet. If it is crawling along the curve $y = x^2$ in the xy-plane, how fast does it have to crawl to outrun the rising tide of the liquid? The vessel may be considered to be as large as necessary.

The solution to this problem is not difficult, but it requires familiarity with volumes, arc length, the fundamental theorem of calculus, and the chain rule. As such, it is a good review problem for a calculus class. Moreover, the problem can be generalized easily enough for students to explore.

To solve the problem, let us first express the volume of the liquid as a function of time. Since the flow rate is constant, this is simply $V(t) = \rho t$, where we have set the initial time $t_0 = 0$. To find the height in the vessel that this volume corresponds to, we compute the volume $\phi(h)$ of a vessel of height h. An easy way to do this is to use the inverse function $x = \sqrt{y}$ and circular cross-sections. The result is $\phi(h) = \int_0^h \pi(\sqrt{y})^2\, dy = \pi h^2/2$. Therefore, the height of the rising liquid is determined by $\rho t = \pi h^2/2$, or $h(t) = \sqrt{2\rho t/\pi}$. On the other hand, the bug is crawling along the curve $x = \sqrt{y}$, and, assuming that it starts from a height $h_0 > 0$, the distance it covers is given by

$$L(t) = \int_{h_0}^{h(t)} \sqrt{1 + ((\sqrt{y})')^2}\, dy = \int_{h_0}^{h(t)} \sqrt{1 + \frac{1}{4y}}\, dy.$$



Its speed is the derivative with respect to time of $L(t)$ for which we need to use the fundamental theorem of calculus and the chain rule: $dL/dt = (1 + 1/4h)^{1/2}(dh/dt)$. We use implicit differentiation on $\rho t = \pi h^2/2$ to obtain $dh/dt = \rho/(\pi h)$, so that the minimum speed of the bug, necessary to keep pace with the rising liquid, is given by

$$\frac{dL}{dt} = \left(1 + \frac{1}{4h}\right)^{1/2} \frac{\rho}{\pi h} = \left(1 + \frac{1}{4}\sqrt{\frac{\pi}{2\rho t}}\right)^{1/2} \left(\frac{\rho}{2\pi t}\right)^{1/2}.$$

Clearly, as the vessel opens up, the rate at which the liquid rises slows down, giving our poor bug the opportunity to catch its breath. The discontinuity at $t = 0$ is due to the discontinuity of the liquid flow rate, which is piecewise constant since it is zero for $t < 0$.

Of course, this problem can be modified by choosing different functions and flow rates, but as we already mentioned, it is easily generalized. If the flow rate ρ is a function of time, then the volume of the liquid after time t is given by $V(t) = \int_0^t \rho(s)\,ds$ and $dV/dt = \rho(t)$. If the function which generates the vessel is defined by $x = f(y)$, then the function $\phi(h)$ is given by $\int_{h_0}^{h} \pi f^2(y)\,dy$ with $d\phi/dh = \pi f^2(h)$. At time t, the volume of the liquid corresponds to the height of the liquid via $V(t) = \phi(h)$, which means that $dV/dt = (d\phi/dh)(dh/t)$. Since $L(t) = \int_{h_0}^{h(t)} \sqrt{1 + f'^2(y)}\,dy$, we obtain

$$\frac{dL}{dt} = \frac{dL}{dh}\frac{dh}{dt} = \sqrt{1 + f'^2(h)}\,\frac{dV/dt}{d\phi/dh} = \frac{\sqrt{1 + f'^2(h)}}{\pi f^2(h)}\,\rho(t).$$

In other words, the speed is the product of a geometric quantity and the flow rate of the liquid. The speed can also be expressed as an explicit function of time, provided we can solve $V(t) = \phi(h)$ explicitly for h.

More variants of this problem can be obtained by defining the revolving function differently, by computing the volume with a different technique, etc. One favorite function of mine is $y = -1/\sqrt{x}$ because revolving that curve, e.g., for $0 < x \leq 1$, around the y-axis generates an infinite vessel of finite volume.

Streaks and Generalized Fibonacci Sequences

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While prolific rabbits may have been the inspiration for Fibonacci when he introduced his renowned sequence, mathematicians, both professional and amateur, find it in many other situations. In this note, we look at how it and a generalization arise in the number of strings of n events having k straight successes.

More formally, we call a sequence of n trials whose outcomes are either successes (S) or failures (F) an *n-string*, and k successes in a row we call a *k-streak*. (Note that streaks only refer to successes.) It turns out that Fibonacci's famous sequence of numbers f_n appears in a formula for the number of 2-streaks in n-strings. Furthermore, there are other Fibonacci-like sequences that appear in the counting of strings with k streaks.

We let $S(n, k)$ denote the number of n-strings that contain a k-streak and $F(n, k)$ the number of these in which the only k-streak occurs at the end. For example, there are eight 4-strings with a 2-streak, but in only two of these, *SFSS* and *FFSS*, is there no 2-streak until the end.

Note that for $n < k$, $S(n, k) = 0$, while for $n = k$, $S(n, k) = 1$. Table 1 shows some other values of $S(n, k)$.

Table 1. Numbers of strings with streaks

n	2	3	4	5	6	7
1	0	0	0	0	0	0
2	1	0	0	0	0	0
3	3	1	0	0	0	0
4	8	3	1	0	0	0
5	19	8	3	1	0	0
6	43	20	8	3	1	0
7	94	47	20	8	3	1
8	201	107	48	20	8	3
9	423	238	111	48	20	8

Focusing our attention on the first column, in looking for a pattern in these numbers—in particular, a relationship between consecutive entries—we find this: