$n = ab(a^2 - b^2)/4$. It is straightforward to check that either choice makes n a sum-difference number. We have proven the following:

Theorem. *A positive integer n is a sum-difference number if and only if there exists a pair of relatively prime numbers a, b with $a > b$ and a positive integer k such that either $n = k^2ab(a^2 - b^2)$ (if one of a or b is even), or $n = k^2ab(a^2 - b^2)/4$ (if both a and b are odd).*

It is an easy exercise to find the values of a, b and k for $n = 6$, $n = 24$, and $n = 30$. Note that a number n can have several factorizations as a sum-difference number, and furthermore, a single such factorization can arise from *different* choices of a, b, and k. For example,

$$\begin{aligned} 210 &= 35 \cdot 6 = 14 \cdot 15 && \text{with } k = 1, a = 5, b = 2, \\ &= 35 \cdot 6 = 14 \cdot 15 && \text{with } k = 1, a = 7, b = 3, \\ &= 42 \cdot 5 = 7 \cdot 30 && \text{with } k = 1, a = 6, b = 1. \end{aligned}$$

An interesting project for students is to study what sets of relatively prime numbers produce the same sum-difference number, or even more, the same factorization as shown in the example above.

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Summary. Starting with an interesting number game sometimes used by school teachers to demonstrate the factorization of integers, *sum-difference numbers* are defined. A positive integer n is a sum-difference number if there exist positive integers x, y, w, z such that $n = xy = wz$ and $x - y = w + z$. This paper characterizes all sum-difference numbers and student exercises and projects are also suggested.

Animating Nested Taylor Polynomials to Approximate a Function

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The conventional method of demonstrating the power of Taylor polynomials is to *increase* the degree and show that the function is better approximated. Here to see, instead, how well Taylor polynomials of a *fixed* degree can approximate a function, we move the center point and superimpose the graphs on one plot. This makes for a lively, action-filled animation that draws students' attention and reinforces the concept. The animations (see [**1**]) can be shown in class or incorporated into a laboratory module. We also show that for a nice class of functions, the graphs of the Taylor polynomials form a nested family and discuss how this can lead to a simple derivation of a bound on the remainder.

Nested families Two Taylor polynomials, T_s and T_t, are part of a nested family if one always lies below the other, that is, they don't intersect (see Figure 1). By differentiating with respect to the center point we obtain a condition for nesting to occur.

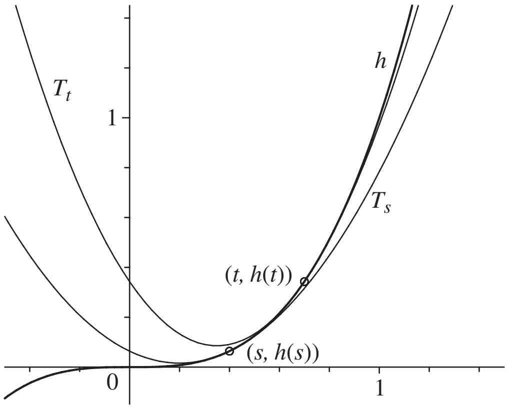

Figure 1. Nesting quadratic Taylor polynomials for $h(x) = x^3$.

Theorem. *Suppose $h : R \to R$ is C^{n+1} and $h^{(n+1)}(w) > 0$ for all w. Let T_w be the nth degree Taylor Polynomial of h about w and let $t > s$. If n is even, then $T_t(x) - T_s(x) > 0$ for all x.*

Proof. Let T_w be the Taylor polynomial of h, centered about an arbitrary point w,

$$T_w(x) = h(w) + h'(w)(x-w) + \frac{h''(w)(x-w)^2}{2!} + \cdots + \frac{h^{(n)}(w)(x-w)^n}{n!}.$$

Differentiating with respect to w,

$$\frac{\partial}{\partial w} T_w(x) = h'(w) - h'(w) + h''(w)(x-w) - h''(w)(x-w) + \cdots - \frac{h^{(n)}(w)(x-w)^{n-1}}{(n-1)!} + \frac{h^{(n+1)}(w)(x-w)^n}{n!},$$

so

$$\frac{\partial}{\partial w} T_w(x) = \frac{h^{(n+1)}(w)(x-w)^n}{n!}. \tag{1}$$

Integrating gives,

$$\int_s^t \frac{h^{(n+1)}(w)(x-w)^n}{n!} dw = T_t(x) - T_s(x). \tag{2}$$

Thus, if n is even and $h^{(n+1)}(w) > 0$ for all w, the integrand is positive for $x \neq w$, and so $T_t(x) - T_s(x) > 0$ for all x. ■

This shows that $y = T_w(x)$ sweeps out a nested family of graphs as w varies. Figure 2 gives an interesting example, where we use second degree Taylor polynomials for a function h with h''' everywhere positive. In the figure, however, the function h is not plotted. Starting in the upper right corner and going counterclockwise, as more Taylor polynomials are included, the actual function is swept out. Providing more and more points animates the family of nested Taylor polynomials approximating the given function. Notice that in the last figure, the graph of the function, $h(x) = e^x$, appears as a thick dark band, even though it is not drawn, providing another way to visualize how Taylor polynomials approximate a function locally.

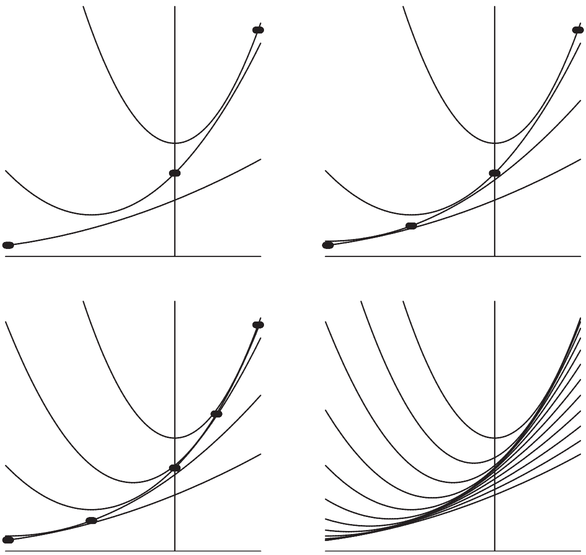

Figure 2. Nesting behavior of 13 evenly spaced second order Taylor polynomials approximating the function, $h(x) = e^x$. The boxes mark the point of expansion in three of the figures.

A second example is in Figure 3. The thick band is not an artifact of the rendering but is caused by the behavior of the Taylor polynomials. Their graphs fit together to create the band because they all approximate the same function locally.

The error bound Derivation of the error bound can cause difficulty for students in a standard calculus course. If an instructor wishes to derive it, equation (2) provides a convenient starting point.

Corollary. *Suppose $h : R \to R$ is C^{n+1} and $|h^{(n+1)}(w)| \le M$ for w between x and s, then*

$$|h(x) - T_s(x)| \le M \frac{|x-s|^{n+1}}{(n+1)!}.$$

Proof. From equation (2) and taking $t = x$, it follows that

$$\int_s^x \frac{h^{(n+1)}(w)(x-w)^n}{n!} dw = T_x(x) - T_s(x) = h(x) - T_s(x).$$

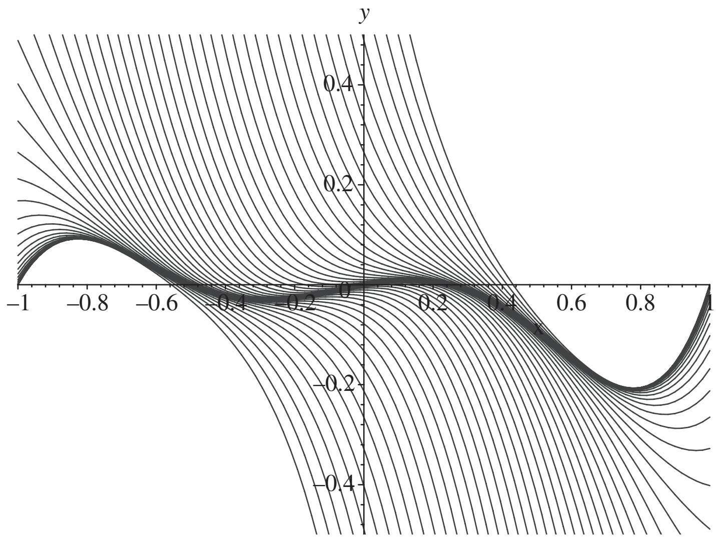

Figure 3. Nesting 4th Taylor polynomials for a fifth degree polynomial.

So, if $|h^{(n+1)}(w)| \leq M$ for w between x and s, then

$$\begin{aligned}|h(x) - T_s(x)| &= \left| \int_s^x h^{(n+1)}(w) \frac{(x-w)^n}{n!} dw \right| \\ &\leq \left| \int_s^x \left| h^{(n+1)}(w) \frac{(x-w)^n}{n!} \right| dw \right| \\ &\leq \left| M \int_s^x \left| \frac{(x-w)^n}{n!} \right| dw \right| = M \frac{|x-s|^{n+1}}{(n+1)!}. \qquad \blacksquare\end{aligned}$$

If n is odd, and $h^{(n+1)}(x) > 0$ for all x, it can be shown that for $t > s$, $T_t(x) - T_s(x)$ is an increasing function of x, and crosses the x-axis between t and s. For degree 1, this means that the tangent lines at s and t intersect exactly once between s and t, if $h''(x) > 0$, illustrating the general notion of convexity.

Summary. The way that Taylor polynomials approximate functions can be demonstrated by moving the center point while keeping the degree fixed. These animations are particularly nice when the Taylor polynomials do not intersect and form a nested family. We prove a result that shows when this nesting occurs. The animations can be shown in class or incorporated into computer labs.

Reference

1. Eric F. Mazzone and Bruce R. Piper, Animations of Taylor polynomials with Moving Centers; available at http://www.rpi.edu/~piperb/taylor.

