The Algebra of Block Permutations

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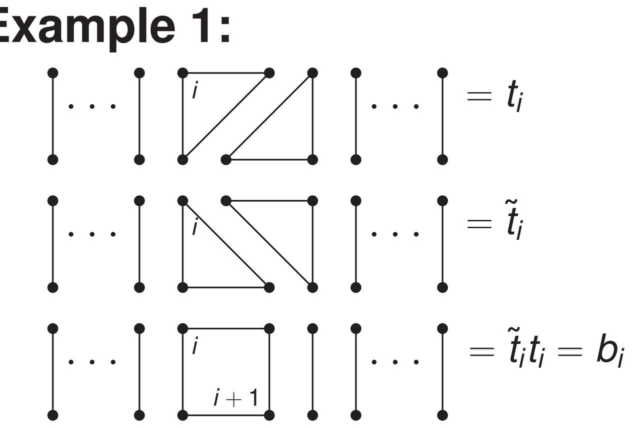
Abstract

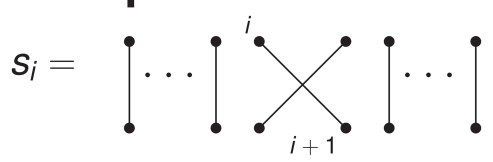
We show that the **set of block permutations of** [**n**], denoted $\mathbf{BP_n}$, is closed under multiplication, and is therefore a monoid. We study the structure of $\mathbf{BP_n}$ as an algebra and the subalgebra of planar diagrams, $\mathbf{P_n}$, give presentations for both, and investigate their representation theory.

Background

- A **set partition** of $[n] = \{1, 2, \ldots, n\}$ is a collection of non-empty disjoint subsets of $[n]$, called **blocks**, whose union is $[n]$.
- A **block permutation** of $[n]$ is a bijection $f : \mathcal{A} \to \mathcal{B}$ between two set partitions $\mathcal{A}, \mathcal{B} \vdash [n]$ both having k blocks, and can be represented as a diagram as follows.
  - **Example 1:**



  - **Example 2:**



- **Example 1** depicts diagram multiplication i.e. $\tilde{t}_i t_i = b_i$.
- g is **planar** if g's diagram can be drawn without edges crossing inside of the rectangle formed by its vertices. In **Example 2**, s_i is non-planar.
- The symmetric group S_n is contained in BP_n.

The Monoid Algebra BP_n

- **Proposition:** The dimension of BP_n is

$$|BP_n| = \sum_{k=1}^{n} k!\,S(n,k)^2,$$

where $S(n,k)$ is the Stirling number of the second kind.

- **Proposition:** The dimension of the planar subalgebra P_n is

$$|P_n| = \sum_{k=1}^{n-1} \binom{n-1}{k-1}^2 = \binom{2(n-1)}{n-1}$$

- This is also the dimension of the planar rook algebra.

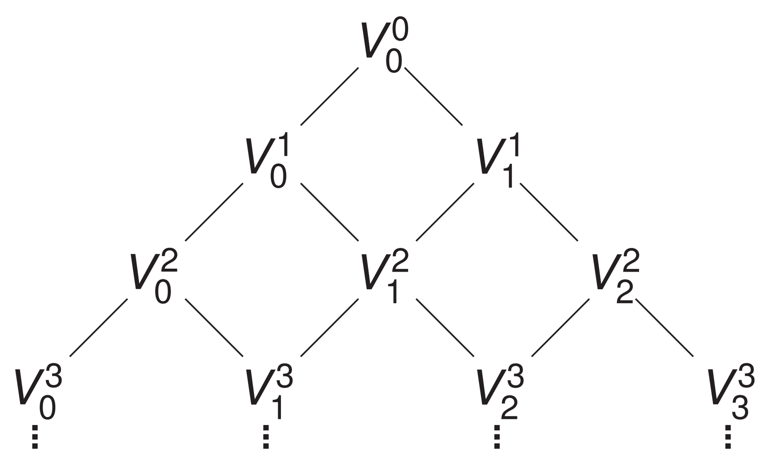
Presentation of P_n

- **Proposition:** P_n is generated by $t_i, \tilde{t}_i, b_i$ as in **Example 1**.
- **Proposition:** $P_n = \langle b_i, t_i, \widetilde{t}_i \mid \mathcal{R} \rangle$, where $\mathcal{R}$ is the following set of relations. They are defined for $1 \le i \le n$ unless otherwise noted.
  - $b_j t_i = t_i b_j$ if $j < i$ or $j - i > 1$
  - $t_i t_j = t_j t_i$ for $|i-j| > 1$
  - $t_i^2 = b_i t_i = t_i b_{i+1} = b_i b_{i+1}$
  - $\tilde{t}_i t_{i+1} = b_{i+2}\tilde{t}_i = \tilde{t}_i b_{i+2}$
  - $t_{i+1}\tilde{t}_i = b_i t_{i+1} = t_{i+1} b_i$
  - $t_{i+1} t_i t_{i+1} = t_i t_{i+1} t_i = t_i t_{i+1}$
  - $\tilde{t}_i t_i = b_i^2 = b_i$
  - $t_i \tilde{t}_i = b_{i+1}$
  - $b_{i+1} t_i = t_i b_i = t_i$

The Planar Rook Algebra PR_n

We show that the **planar rook algebra** $\mathbf{PR_n}$, which is the set of $n \times n$ matrices with entries from $\{0, 1\}$ having at most one 1 in each row and column, of Halverson et al. is isomorphic to the planar subalgebra P_{n+1}. Consequently, we can classify the irreducible representations of P_n using the results of [3].

- **Theorem:** $PR_n \cong P_{n+1}$.
- **Proposition:** RP_n acts on $V^n = \text{span}_{\mathbb{C}}\{v_S \mid S \subset \{1, \ldots, n\}\}$ and $V^n \cong \bigoplus_{k=0}^{n} V_k^n$, where $V_k^n = \text{span}_{\mathbb{C}}\{v_S \mid S \subset \{1, \ldots, n\}, |S| = k\}$ and each submodule has multiplicity 1.
- We have the following Bratteli Diagram. Lines represent the restriction of irreducible representations of RP_n to those of RP_{n-1}, which arises from the fact that $V_k^n \cong V_{k-1}^{n-1} \oplus V_k^{n-1}$ as an RP_{n-1} module.



- The dimensions of each module is given by $\binom{n}{k}$. If we replace each module with its corresponding dimension, we obtain Pascal's triangle.
- **Proposition:** For $0 \le k \le n$ and $f \in P_{n+1}$, the irreducible characters are given by

$$\chi_k^n(f) = \begin{cases} \binom{i}{k} & \text{if } k \le i \\ 0 & \text{if } k > i, \end{cases}$$

where i is the number of vertical edges in $d \in RP_n$.
- The character table for χ_k^n, the irreducible character of V_k^n, is given by Pascal's triangle.

Presentation of BP_n

- **Proposition:** BP_n is generated by $t_i, \widetilde{t}_i, b_i$, and the elementary transpositions s_i.
- **Proposition:** $BP_n = \langle s_i, b_i, t_i, \widetilde{t}_i \mid \mathcal{R}' \rangle$, where $\mathcal{R}'$ consists of $\mathcal{R}$ and the following relations:
  - $s_i s_j = s_j s_i$ if $|i-j| > 1$
  - $b_i s_j = s_j b_i$ if $|i-j| > 1$
  - $t_i s_j = s_j t_i$ if $j - i > 2$ or $i - j > 1$
  - $t_{i-1} s_i t_i = s_{i-1} s_i b_{i-1} b_{i+1} s_i s_{i-1}$
  - $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
  - $s_i b_{i+1} s_i = s_{i+1} b_i s_{i+1}$
  - $s_{i+2} t_i s_{i+2} = s_{i+1} t_{i+1} t_i$
  - $t_i = t_i s_i = s_{i+1} t_i$
  - $s_i^2 = 1$
  - $s_i t_{i+1} s_i = t_{i+1} t_i s_{i+1}$
- **Proposition:** BP_n factors as $BP_n = S_n P_n S_n$.
- **Example:** $t_1 s_2 s_1 t_1 = s_1 s_2 b_1$.

Representation Theory of BP_n

- **Proposition:** $I = BP_n b_{n-1} BP_n = \{f \cdot b_{n-1} \cdot g \mid f, g \in BP_n\}$ is an ideal of BP_n.
- **Corollary:** $BP_n \cong BP_n b_{n-1} BP_n \oplus \mathbb{C}[S_n]$.
- **Proposition:** $BP_{n-1} \cong b_{n-1} BP_n b_{n-1}$.
- We use the above propositions to show that the irreducible representations of BP_n can be indexed by Young Diagrams.

Shuffle Product and Breaking Points

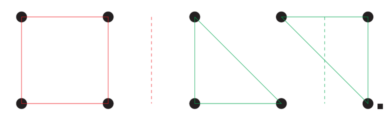
- A $(\mathbf{p}, \mathbf{q})$**-shuffle** is a permutation $\xi \in S_{p+q}$ such that

$$\xi(1) < \cdots < \xi(p)$$
and
$$\xi(p+1) < \cdots < \xi(p+q).$$

- **Example 3:** The following are S_{1+2} shuffles.

$$\xi_1 = \begin{pmatrix} 1 & 2\,3 \\ 1 & 2\,3 \end{pmatrix} \quad \xi_2 = \begin{pmatrix} 1 & 2\,3 \\ 2 & 1\,3 \end{pmatrix} \quad \xi_3 = \begin{pmatrix} 1 & 2\,3 \\ 3 & 1\,2 \end{pmatrix}$$

- Let $f \in BP_p$ and $g \in BP_q$. We obtain the **concatenation** of f and g, $\mathbf{f} \times \mathbf{g} \in BP_{p+q}$, by adding p to every vertex in g and placing g to the right of f.
- **Example 4:** If $b_1 \in BP_2$ and $\tilde{t}_1 \in BP_3$, then $b_1 \times \tilde{t}_1 \in BP_5$

$b_1 \times \tilde{t}_1 =$ 

- $i \in [n]$ is a **breaking point** of $f \in BP_n$ if one can place a vertical line between the vertices i and $i+1$, and if the sum of the sizes of blocks of $\mathcal{A}$ mapping to blocks of $\mathcal{B}$ up to i equals i. We denote the set of breaking points of f by $B(f)$.
- In **Example 4**, the red line at $i = 2$ is a breaking point, but the green one at $i = 4$ is not.

Hopf Algebra Structure of BP_n

- **Theorem:** The graded vector space

$$\mathcal{BP} = \bigoplus_{n \ge 0} \mathbb{K}[BP_n]$$

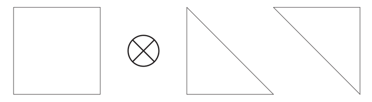
with the product $*$, coproduct Δ, unit $\emptyset$, and counit ϵ is a **graded connected Hopf algebra**.
- The **unit** is $\emptyset$, the empty diagram, and the **counit** is the map $\epsilon : \mathcal{BP} \to \mathbb{K}$ given by $\epsilon(f) = \delta_{\emptyset, f}$.
- We define the **product** $*$ on $\mathcal{BP}$ to be

$$f * g = \sum_{\xi} \xi \cdot (f \times g) \in \mathbb{K}[BP_{p+q}]$$

$\forall\, f \in BP_p,\ g \in BP_q$ and all (p,q)-shuffles ξ.
- If we take ξ_1, ξ_2, ξ_3 as in Example 3, then the shuffle product of two diagrams $f \in BP_1$ and $g \in BP_2$ is just the diagram multiplication of $f \times g$ distributed over $\xi_1 + \xi_2 + \xi_3$.
- We define the **coproduct** on $\mathcal{BP}$ to be

$$\Delta(f) = \sum_{i \in B(f)} f_{(i)} \otimes f'_{(n-i)}.$$

- Let $f = b_1 \times \tilde{t}_1$ as in **Example 4**. Then $\Delta(f)$ is given by

$\Delta(f) = \emptyset \otimes f +$  $+ f \otimes \emptyset$.

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