$\sum a_n$ and $\sum 2^m a_{2^m}$ converge or diverge together; just group the terms of $\sum a_n$ into dyadic blocks to prove it [**2**]. For our second proof: we let

$$a_n = \frac{1}{n \log_2 n \cdots \log_2^{k-1} n (\log_2^k n)^p}.$$

Then $\log_2^j 2^m = \log_2^{j-1}\left(\log_2(2^m)\right) = \log_2^{j-1} m$ and

$$2^m a_{2^m} = 2^m \frac{1}{2^m m \cdots \log_2^{k-2} m (\log_2^{k-1} m)^p},$$

so since the terms of S are eventually decreasing, the k case is equivalent to the $k-1$ case, is equivalent to the $k-2$ case, ..., is equivalent to the $k = 0$ case. Thus $S(2, k, p)$ converges if and only if $\sum n^{-p}$ converges. A final application of the Cauchy Condensation Test reduces this to the easy analysis of the geometric series $\sum (2^{1-p})^m$. ■

Some history. The treatment of the natural logarithm cases $S(e, k, p)$ by means of the Integral Test is very well known. For example, they are treated in the classic encyclopedic work of Konrad Knopp, where 4 distinct proofs are given, one of which uses Cauchy's Condensation Test. That particular proof can easily be modified to do the cases $S(b, k, p)$ when $b \geq 2$ but not the cases when $1 < b < 2$ [**3**].

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Sums of Integer Powers via the Stolz-Cesàro Theorem

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One of the powerful rules for evaluating limits of sequences or series is the Stolz-Cesàro theorem which is a discrete form of l'Hôpital's rule.

Stolz-Cesàro Theorem. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. If b_n is positive, strictly increasing and unbounded and the following limit exists:*

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = l,$$

then the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l$$

also exists and it is equal to l.

It has been known for years that the sum of integer powers $S_k(n) = \sum_{i=1}^{n} i^k$ is a polynomial of degree $k+1$ in the variable n having rational coefficients [**1–5**]. For example, it is well known that

$$1 + 2 + \cdots + n = \frac{1}{2}n^2 + \frac{1}{2}n.$$

In this note, we illustrate the power of the Stolz-Cesàro theorem by using it to determine the coefficients of this polynomial.

Suppose

$$S_k(n) = 1^k + 2^k + \cdots + n^k = c_{k+1}n^{k+1} + c_k n^k + \cdots + c_1 n + c_0,$$

where $c_0 = 0$. To get c_{k+1}, let us divide the above expression by n^{k+1}, then take the limit (using the Stolz-Cesàro theoem, since n^{k+1} is strictly increasing) as $n \to \infty$.

$$\begin{aligned} c_{k+1} &= \lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k - c_k n^k - \cdots - c_1 n}{n^{k+1}} = \lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} \\ &= \lim_{n\to\infty} \frac{n^k}{n^{k+1} - (n-1)^{k+1}} = \frac{1}{k+1}. \end{aligned} \tag{1}$$

For $j = 1, 2, \ldots, k$,

$$c_j = \frac{S_k(n) - c_{k+1}n^{k+1} - c_k n^k - \cdots (\text{omitting } c_j n^j) \cdots - c_1 n - c_0}{n^j}$$

for all n. Therefore,

$$\begin{aligned} c_j &= \lim_{n\to\infty} \frac{S_k(n) - c_{k+1}n^{k+1} - c_k n^k - \cdots - c_{j+1}n^{j+1}}{n^j} \\ &= \lim_{n\to\infty} \frac{n^k - c_{k+1}(n^{k+1} - (n-1)^{k+1}) - c_k(n^k - (n-1)^k) - \cdots - c_{j+1}(n^{j+1} - (n-1)^{j+1})}{n^j - (n-1)^j}. \end{aligned}$$

The highest order term in the denominator is jn^{j-1}. We know that the limit exists, so all of the terms of order higher than $j-1$ in the numerator must vanish, all of the terms of lower order have no effect on the limit. Expanding the terms in the numerator and simplifying, we find that the $(j-1)$th order term is

$$\left[(-1)^{k-j}c_{k+1}\binom{k+1}{j-1} + (-1)^{k-j-1}c_k\binom{k}{j-1} + \cdots + c_{j+1}\binom{j+1}{j-1}\right]n^{j-1}.$$

Consequently,

$$c_j = \frac{(-1)^{k-j}c_{k+1}\binom{k+1}{j-1} + (-1)^{k-j-1}c_k\binom{k}{j-1} + \cdots + c_{j+1}\binom{j+1}{j-1}}{j}. \tag{2}$$

Now by applying (1) and (2) we can compute $c_{k+1}, c_k, \ldots, c_0$ recursively. For example,

k	c_{k+1}	c_k	c_{k-1}	$\cdots$	$S_k(n)$
0	1				n
1	$\frac{1}{2}$	$\frac{1}{2}$			$\frac{1}{2}n^2 + \frac{1}{2}n$
2	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$		$\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

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Short Division of Polynomials

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When I teach, I always tell my students to ask me "why". So when I was teaching synthetic division in precalculus, my student Heysel Marte promptly asked why the algorithm worked. I told her that it was an excellent question, thought for a moment, and then presented the following sketch to the class.

"Suppose that we want to divide $ax^3 + bx^2 + cx + d$ by $x - k$. Then we are seeking an answer of the form

$$ax^3 + bx^2 + cx + d = (ex^2 + fx + g)(x - k) + r.$$

Expanding the right-hand side we get $ex^3 + (f - ek)x^2 + (g - fk)x + r - gk$. Comparing coefficients we see that $e = a$, $f = b + ek$, $g = c + fk$, and $r = d + gk$, which is exactly what synthetic division is doing."

$$\begin{array}{c|cccc}
k & a & b & c & d \\
 & & ek & fk & gk \\
\hline
 & e & f & \multicolumn{1}{c|}{g} & r
\end{array}$$

As a casual comment, I pointed out the caution in the textbook about the limitations of the algorithm, but after the class, I asked myself why the idea above could not be extended to divisors of higher degrees. Then I realized that it COULD!

Suppose that we want to divide $ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$ by $x^2 - kx - l$. Then we are seeking an answer of the form

$$\begin{aligned} &ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g \\ &\quad = (mx^4 + nx^3 + ox^2 + px + q)(x^2 - kx - l) + (rx + s). \end{aligned}$$

Expanding the right-hand side and comparing coefficients we see that $m = a$, $n = b + mk$, $o = c + nk + ml$, $p = d + ok + nl$, $q = e + pk + ol$, $r = f + qk + pl$, and $s = g + ql$. We now display these relations in a format of synthetic division.

$$\begin{array}{cc|ccccccc}
k & l & a & b & c & d & e & f & g \\
 & & & mk & nk & ok & pk & qk & \\
 & & & & ml & nl & ol & pl & ql \\
\hline
 & & m & n & o & p & \multicolumn{1}{c|}{q} & r & s
\end{array}$$

