Inequalities of the Form $f(g(x)) \geqslant f(x)$

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Consider a problem in trigonometry [1]: To prove that for all x in the interval $[0, \pi]$,

$$(\sin x)(1+\cos x) \leqslant \left[\sin\left[\frac{x+\pi}{4}\right]\right]\left[1+\cos\left[\frac{x+\pi}{4}\right]\right]. \tag{1}$$

It is possible to solve this problem by manipulating both sides using trigonometric identities, but this method leads to complicated expressions which yield little insight into the problem. A different approach is to reformulate the problem in functional terms: It is the same as proving that

$$F(G(x)) \geqslant F(x), \tag{2}$$

where $F(x) = (\sin x)(1+\cos x)$ and $G(x) = (x+\pi)/4$.

In this note, a simple method is given for constructing functions G such that (2) holds for a given function F and all x in a given closed interval $[a, b]$. The method will then be illustrated in two applications. The related functional inequality $F(G(x)) \geqslant G(F(x))$ is considered in [2].

Let F and G denote continuous functions on a finite closed interval $[a, b]$. If the following general conditions are satisfied, then (2) holds for all x in $[a, b]$:

C1: Interval $[a, b]$ is partitioned into N closed subintervals $\{I_n,\ 1 \leqslant n \leqslant N\}$ such that for each n, F is monotonic on I_n.
C2: For each n, G maps I_n into I_n.
C3: In each subinterval where F increases, $G(x) \geqslant x$. In each subinterval where F decreases, $G(x) \leqslant x$.

Graphically, C2 and C3 imply that for each $n \leqslant N$, the graph of G lies within the square $I_n \times I_n$, above or below the line $y = x$ according to whether F is increasing or decreasing in I_n. This is illustrated in FIGURE 1.

The trigonometric inequality above can be proven using these ideas. As above, let $F(x) = (\sin x)(1+\cos x)$ and $G(x) = (x+\pi)/4$. F is increasing in $[0, \pi/3]$ and decreasing in $[\pi/3, \pi]$. Each linear function $G(x) = Ax + B$ satisfies conditions C2 and C3 provided that $G(\pi/3) = \pi/3$ and $0 \leqslant A \leqslant 1$; the G given in the statement of the problem is a member of a continuous family of linear functions for which the inequality holds.

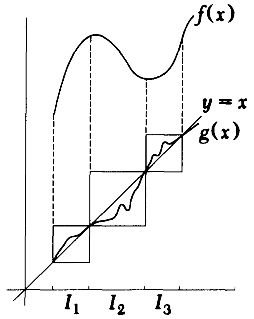

FIGURE 1

The conditions stated above can be used to prove an interesting theorem. Define $P(x) = G(x) - x$. Then C3 implies that sgn $P(x) =$ sgn $F'(x)$ for all x in (a, b). This suggests that there should exist a 'small' positive real constant k such that $G(x) = x + kF'(x)$ would satisfy conditions C2 and C3. However, if $F'(a) < 0$, then this would imply that $G(a) = a + kF'(a) < a$, contradicting condition C2 requiring that G map $[a, b]$ into itself. The same problem would arise if $F'(b)$ were positive. By introducing the non-negative multiplier $(x - a)(b - x)$, this problem is avoided.

THEOREM. *Let F be a twice-differentiable function with a finite number of local extrema in $[a, b]$. Then there exists a positive real constant k such that for all x in $[a, b]$,*

$$F(x) \leqslant F[x + k(x-a)(b-x)].$$

Proof. The finite set of local extrema of F furnishes a subdivision of $[a, b]$ into intervals on which F is monotonic. Consider one such interval $I = [c, d]$ on which F increases. To satisfy C2 and C3, it is sufficient to determine a positive real constant k such that the following inequality is satisfied in I:

$$c \leqslant x \leqslant x + k(x-a)(b-x)F'(x) \leqslant d.$$

The two leftmost inequalities are obvious. The last inequality holds iff

$$k(x-a)(b-x)F'(x) \leqslant d - x.$$

Case 1: $d = b$. The inequality will hold when

$$k(x-a)F'(x) \leqslant 1.$$

The existence of F'' implies continuity and boundedness of F'; let M be the maximum absolute value of F' in $[a, b]$.

$$k(x-a)F'(x) \leqslant k(b-a)M; \quad \text{choose } k = \frac{1}{M(b-a)}.$$

Case 2: $d < b$. Then $F'(d) = 0$ and

$$k(x-a)(b-x)[F'(x) - F'(d)] \leqslant d - x$$

$$\Leftrightarrow \qquad k(x-a)(b-x)[(F'(x) - F'(d))/(d-x)] \leqslant 1.$$

Since the difference quotient approaches $-F'(d)$ as x approaches d, there exists a number t in (c, d) such that the quotient is bounded above in (t, d). Therefore, the absolute value of the difference quotient in $[c, d]$ is bounded above by some positive number M. Also, $(x - a)(x - b) < (b - a)^2$. Take

$$k = \frac{1}{M(b-a)^2}$$

and the inequality is satisfied. A similar argument applies if F decreases on I. Thus the above shows how to chose k in any specific subinterval I. Taking the minimum value of k over all the subintervals of the partition, one obtains a value of k valid for the entire interval $[a, b]$. Q.E.D.

The same argument can be used to establish the existence of a positive real constant h such that for all x in $[a, b]$,

$$F(x) \geqslant F[x - h(x-a)(b-x)F'(x)].$$

If $F'(a) \geqslant 0$, then the multiplier $(x-a)$ in the statement of the theorem is superfluous, and if $F'(b) \leqslant 0$, then $(b-x)$ is superfluous. For example, taking $F(x) = \sin x$ and $[a, b] = [0, \pi]$, we find that G can be of the form $G(x) = x + kF'(x)$. Applying to this pair of functions the argument used to prove the theorem, we find that for $0 \leqslant x \leqslant \pi$,

$$0 \leqslant k \leqslant 1 \Rightarrow \sin(x + k(\cos x)) \geqslant \sin x.$$

Setting $x = \pi/2 - y$, we obtain the equivalent inequality

$$\cos(y - k(\sin y)) \geqslant \cos y, \quad -\pi/2 \leqslant y \leqslant \pi/2.$$

Other inequalities of similar form may be written down by varying the function or the interval.

REFERENCES

1. *The Arbelos*, March 1987, p. 11, problem 20.
2. Boas, R. P., Inequalities for a Collection, this MAGAZINE, 52 (Jan. 1979), 29–31.

Poetry Analysis

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While watching the classic film, *Singing in the Rain*, recently, we were intrigued by the curious behavior of the protagonist in the song whose lyrics are

Moses supposes his toeses are roses,
But Moses supposes erroneously.
Moses he knowses his toeses aren't roses
As Moses supposes his toeses to be.

Upon analysis, we concluded that there are only two[1] reasonable explanations for the thought patterns attributed to Moses in this poem.

a) Moses is suffering from an obsessive-compulsive disorder, which compels him to irrational beliefs even though he is aware of their irrationality, or[2]
b) Moses is a mathematician who is attempting to prove by contradiction that his toeses are not roses.

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[1]After briefly considering a multiple personality as an additional possibility, we decided that, if this were the case, the third line would be

Abraham knowses his toeses aren't roses.

[2]As usual, this is the inclusive 'or.'