A Single Inequality Condition for the Existence of Many r-gons

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In [1] it was found that if $a_1, a_2, \ldots, a_n$ are positive numbers with $n \geqslant 3$, then a sufficient condition that every three of them are the lengths of sides of a triangle is given by

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^2 > (n-1)\left(a_1^4 + a_2^4 + \cdots + a_n^4\right).$$

It was left as an open problem to find a *single* inequality condition on the above n numbers which is both a necessary and sufficient condition that any r of them are the lengths of sides of an r-gon ($n \geqslant r \geqslant 3$).

If the a_i's were ordered, i.e., $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, it then follows easily that the desired inequality condition is

$$a_1 + a_2 + \cdots + a_{r-1} > a_n. \tag{1}$$

However, this really involves a total of n inequalities. We now show how to obtain a single inequality condition, albeit rather complicated, that is equivalent to the latter n inequalities.

Let C_i denote any one of the $\binom{n}{r}$ combinations of r terms $x_1, x_2, \ldots, x_r$ from the n terms $a_1, a_2, \ldots, a_n$. Then a necessary and sufficient condition that the x_i's are lengths of sides of an r-gon is

$$P_i \equiv \Pi(S - 2x_j) > 0,$$

where $S = x_1 + x_2 + \cdots + x_r$. It now follows that a necessary and sufficient condition that every combination of r terms from $a_1, a_2, \ldots, a_n$ are lengths of sides of an r-gon is simply

$$\min P_i > 0,$$

where i is over all $\binom{n}{r}$ combinations.

Finally, we show how to obtain an explicit formula for the latter inequality in terms of the absolute value function. Let

$$M_{r+1} = M(M_r, P_{r+1}), \qquad r = 1, 2, \ldots, \binom{n}{r} - 1,$$

where $M_1 = P_1$ and $M(a,b) = (|a+b| - |a-b|)/2$. Then, $M_2 = \min(P_1, P_2)$, $M_3 = \min(P_1, P_2, P_3), \ldots$. Thus our single inequality condition is

$$M_s > 0, \quad \text{where } s = \binom{n}{r}.$$

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It is still an open problem whether or not there exists a single polynomial inequality that does the job. Most likely, there is not.

REFERENCE

1. M. S. Klamkin, Simultaneous triangle inequalities, this MAGAZINE 60 (1987), 236–237.

A Formula Yielding an Approximate Solution for Some Higher Degree Trinomial Equations

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This note will offer a formula that yields a close approximation of one root for a limited class of higher degree equations, those that can be expressed in the form, $10^N = 10^V X^A + 10^W X^B$, with N, V, and W any real numbers and with A and B both having the same sign and absolute values $\geqslant 1$. It yields accurate values for positive or negative A and B and large or small values of 10^N.

To begin, we pretend that $10^W X^B$ has a value of zero, so that $10^N = 10^V X^A$, and $X = 10^{(N-V)/A}$. If we retain this value for X and then acknowledge $10^W X^B$ as a positive quantity, we get $10^W X^B = 10^{B(N-V)/A+W}$, and $10^W X^B / 10^V X^A = 10^{B(N-V)/A+W-N}$. We will call this ratio 10^C, so $10^N = 10^V X^A(1+10^C)$.

The factor $(1+10^C)$ by which 10^N became enlarged in the above process represents the magnitude of the error caused by our initially ignoring the $10^W X^B$ term. To find the correct value of X, we must divide the X derived from $10^N = 10^V X^A$ by some power of $(1+10^C)$ so that, when X is entered into the full equation that includes a nonzero $10^W X^B$ term, 10^N will no longer be enlarged. That power will be closer to $1/A$ or $1/B$ depending on which term, $10^V X^A$ or $10^W X^B$, respectively, is more influential. We will use the ratio 10^C to determine how far along the gap $(1/A - 1/B)$ we should move, starting at $1/B$. Our formula is thus

$$X \doteq \frac{10^{(N-V)/A}}{(1+10^C)^{1/B+(1/A-1/B)[1/(1+10^C)]}},$$

where $C = B(N-V)/A + W - N$.

Accuracy A comparison between the formula's solutions and the first value of X a computer finds yielding a 10^N differing with the original 10^N by less than 0.001% reveals that, whether A and B are negative or positive, the formula is least accurate when $1 \leqslant N \leqslant 2$. In this range, the formula's error can be as high as 10%, but as N is increased or decreased, the accuracy improves. For $3 \leqslant N \leqslant 4$ or $-2 \leqslant N \leqslant -1$, the maximum error is less than 1%, and for $6 \leqslant N \leqslant 7$ or $-5 \leqslant N \leqslant -4$, less than 0.1%.