Finding the Volume of an Ellipsoid Using Cross-Sectional Slices

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While I was consulting at a major medical complex, a medical researcher asked how he could find the volume of an object from cross-sectional slices through the object. (What actually happened was that foreign objects were imbedded in the lungs of a rat and, after some appropriate time period, a section of lung tissue was removed. Cross-sections were taken of the extracted section and some of the cross-sections contained slices through the mass formed by the body of the rat as it encased the implanted object. The researcher wanted to know the volume of the mass formed by the body when the section of tissue extracted contained only a portion of the mass.) However, the cross-sections of the mass appeared to be elliptical, and the researcher indicated that an ellipsoidal shape for the entire mass was expected. Thus I was led to consider (separate from this particular problem) whether the volume of an ellipsoid can be found exactly from some minimum number of parallel cross-sections. The purpose of this paper is to show how the volume can be calculated from three equally spaced slices.

Assume an ellipsoid is positioned in three-dimensional space with the orientation of the ellipsoid unknown. Axes may be chosen so that the ellipsoid is centered at the origin, slices through the ellipsoid are made parallel to the xy-plane, and the x and y axes lie along the principal axes of the equatorial ellipse cut by $z=0$. Thus, without loss of generality, we may suppose that the ellipsoid has equation

$$X^tAX = 1 \tag{1}$$

where $X^t=(x,y,z)$ and $A=\begin{bmatrix}\lambda_1 & 0 & p\\ 0 & \lambda_2 & q\\ p & q & c\end{bmatrix}$, $\lambda_i>0$. Rewriting (1) for $z=k$ leads to

$$\lambda_1\left(x+\frac{pk}{\lambda_1}\right)^2+\lambda_2\left(y+\frac{qk}{\lambda_2}\right)^2 = 1-k^2c+\frac{p^2k^2}{\lambda_1}+\frac{q^2k^2}{\lambda_2} \tag{2}$$

$$= 1-k^2|A|/\lambda_1\lambda_2 \tag{3}$$

where $|A|=c\lambda_1\lambda_2-q^2\lambda_1-p^2\lambda_2$. Thus, whenever $k^2<\lambda_1\lambda_2/|A|$, the cross-section of the ellipsoid at $z=k$ is the ellipse with easily determined semiaxes a and b and area πab centered at $k[-p/\lambda_1, -q/\lambda_2, 1]$. Now let

$$S=|A|/\lambda_1\lambda_2, \tag{4}$$

so that

$$a^2=\frac{1-k^2S}{\lambda_1},\qquad b^2=\frac{1-k^2S}{\lambda_2}\quad\text{and}\quad (ab)^2=\frac{(1-k^2S)^2}{\lambda_1\lambda_2}. \tag{5}$$

As stated earlier, the purpose of this investigation is to determine the volume of the ellipsoid described in (1). It is well-known that an ellipsoid with semiaxes a, b, c has volume

$$V = 4\pi abc/3$$

and that $1/a^2, 1/b^2, 1/c^2$ are the eigenvalues for the symmetric matrix A. Hence

$$V = 4\pi |A|^{-1/2}/3. \tag{6}$$

A process to find the volume of the ellipsoid is now unfolding: The volume can be found from (6) if $|A|$ is known; $|A|$ can be found from (4) if S, λ_1 and λ_2 are known; S, λ_1 and λ_2 can be found from (5) for various values of a, b and k.

As three values must be found from (5), three cross-sectional ellipses need to be available so that (a_i, b_i), $i = 1, 2, 3$ can be used to find S, λ_1 and λ_2. Let $z = k_i$, $i = 1, 2, 3$ represent the values of z for the three slices. If the slices are equally spaced, then we may take $k_i = k_1 + (i-1)\Delta z$, $i = 1, 2, 3$, $\Delta z \neq 0$. Letting (a_i, b_i) be the semiaxes for the ellipse at $z = k_i$, $i = 1, 2, 3$ and using (5), we obtain

$$(a_i b_i)^2 \lambda_1 \lambda_2 = \left(1 - k_i^2 S\right)^2, \qquad i = 1, 2, 3.$$

Solving for $\lambda_1 \lambda_2$ and equating, we further obtain

$$\frac{\left(1 - k_1^2 S\right)^2}{(a_1 b_1)^2} = \frac{\left(1 - k_2^2 S\right)^2}{(a_2 b_2)^2} = \frac{\left(1 - k_3^2 S\right)^2}{(a_3 b_3)^2}. \tag{7}$$

From (4),

$$k^2 < \frac{\lambda_1 \lambda_2}{|A|} = \frac{1}{S}.$$

Therefore, $0 < 1 - k^2 S$, so that taking positive roots of (7) leads to

$$\frac{1 - k_1^2 S}{a_1 b_1} = \frac{1 - k_2^2 S}{a_2 b_2} = \frac{1 - k_3^2 S}{a_3 b_3}.$$

Solving for S, we obtain

$$\frac{a_1 b_1 - a_2 b_2}{a_1 b_1 (k_1 + \Delta z)^2 - a_2 b_2 k_1^2} = S = \frac{a_1 b_1 - a_3 b_3}{a_1 b_1 (k_1 + 2\Delta z)^2 - a_3 b_3 k_1^2} \tag{8}$$

Solving for k_1, we have

$$k_1 = \frac{-\Delta z}{2} \left[\frac{3a_1 b_1 - 4a_2 b_2 + a_3 b_3}{a_1 b_1 - 2a_2 b_2 + a_3 b_3} \right]. \tag{9}$$

Notice that k_1 is determined by information from the three cross-sectional ellipses and Δz, the distance between slices. We can now compute the volume of an ellipsoid from which three equally spaced cross-sectional slices have been taken:

Step 1. Compute $a_1 b_1$, $a_2 b_2$, and $a_3 b_3$, the product of the semiaxes for each ellipse (each product is the area/π). Also note the value of Δz, the distance between slices.

Step 2. Compute k_1 from (9).

Step 3. Compute S from (8).
Step 4. Compute $\lambda_1\lambda_2$ from (7).
Step 5. Compute $|A| = \lambda_1\lambda_2 S$.
Step 6. Compute $V = \frac{4}{3}\pi|A|^{-1/2}$.

Note that it is possible to interchange (a_1, b_1) and (a_3, b_3), but the center slice must be correctly identified.

For example, let $\Delta z = 1$ cm, and suppose measurements of the cross-sections give (in cm)

first slice:	$a_1 = .620,$	$b_1 = 1.460$
second slice:	$a_2 = .920,$	$b_2 = 2.165$
third slice:	$a_3 = 1.000,$	$b_3 = 2.353$

The algorithm gives a volume $V \simeq 25.1$.

REFERENCES

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Representing Primes by Binary Quadratic Forms

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The problem of representing a prime via a binary quadratic form is one of the oldest and richest in number theory. The prototypical result in this subject is the assertion, first stated and perhaps proved by Fermat in 1640 and proved by Euler in the 1740s, that every positive prime congruent to 1 modulo 4 is a sum of two integer squares. Most proofs of this theorem have the following format: First, it is shown that if $p \equiv 1 \pmod 4$, then there exists $x \in \mathbf{Z}$ (the ring of integers) such that $x^2 \equiv -1 \pmod p$; i.e. -1 is a quadratic residue mod p. Then from the existence of the integer x, one deduces that there exist $a, b \in \mathbf{Z}$ such that $p = a^2 + b^2$.

In **[2]** Larson gives a proof of Fermat's statement that is interesting for at least two reasons. First, Larson's argument, which is based on ideas of Kraitchik **[1]** and Pólya **[3]**, does not use the fact that -1 is a quadratic residue mod p. Second, his proof consists of a pleasant blend of algebraic and geometric arguments. The purpose of this paper is to apply Larson's method to prove the following theorem.