Since $n \geqslant 3$, we see that $\cos(\pi/n) > 0$, and so from equation (5) we learn

$$k = [2\sin(\pi/n)]/\ell. \tag{6}$$

As the curvature is constant, we conclude (solve the differential equations in (2)) that C is a circle.

It is interesting to examine the above proof in the case $n = 2$. Here $\rho = i$ and so $\omega = \zeta$ as expected. The derivation that ℓ is a constant remains valid and so we obtain a proof of the result we mentioned in the introduction: if there is a periodic billiard 2-gon at each point of a smooth convex curve C (i.e., if C has the double normal property), then C has constant width. Notice that in this case equation (5) gives no information about the curvature k, since $\cos(\pi/2) = 0$.

References

[1] D. W. DeTemple and J. M. Robertson, A billiard characterization of regular polygons, this MAGAZINE, 54 (1981) 73–75.

[2] Y. G. Sinai, Introduction to Ergodic Theory, Princeton University Press, 1976.

[3] R. Sine and V. Kreĭnovič, Remarks on billiards, Amer. Math. Monthly, 86 (1979) 204–206.

A Mean Value Property of the Derivative of Quadratic Polynomials—without Mean Values and Derivatives

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Quadratic polynomials $f(x) = ax^2 + bx + c$ have the remarkable property that on their graphs (parabolas) the abscissa of the point where the tangent is parallel to a chord is the arithmetic mean of the abscissas of the endpoints of the chord (see FIGURE 1). In formula, this translates to the following particular mean value property of the derivative of quadratic polynomials:

$$\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x+y}{2}\right) \tag{1}$$

for all real $x \neq y$. In fact, this property is *characteristic* to parabolas; that is, *only* $f(x) = ax^2 + bx + c$ (with arbitrary constants a, b, c) satisfies (1) for all x, y. For a proof see, for instance, [**5**, p. 122], where f was supposed to be three times differentiable. Of course, for (1) to make sense, f has to be differentiable. Then, by (1), it is also twice, three times, actually any number of times differentiable.

On the other hand, (1) is a special case of the equation

$$\frac{f(x) - f(y)}{x - y} = h(x+y) \qquad (x \neq y;\, x, y \in \mathbb{R}) \tag{2}$$

which contains no derivative (and no mean value). So here it does not seem natural anymore to suppose that f (or h) is differentiable. Such equations, which serve to determine unknown functions and don't contain derivatives and integrals, are called *functional equations*. A remarkable feature of functional equations is that one equation can determine several unknown functions, in this case both f and h.

I proved *without any* differentiability or other *regularity conditions* some twenty years ago that the equation (2), containing two unknown functions, also has $f(x) = ax^2 + bx + c$ and $h(x) = ax + b$ as its only solutions. Having done this, I soon forgot about it, but both the result and the

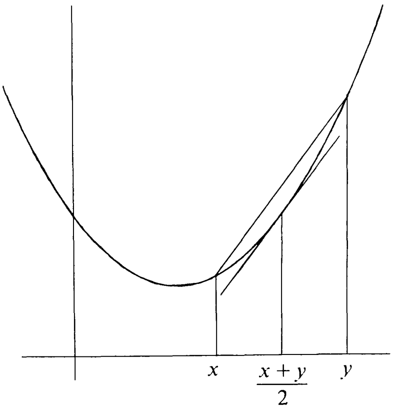

FIGURE 1

proof were preserved in [**1**]. The proof made use of the so-called Jensen functional equation

$$\phi\left(\frac{x+y}{2}\right)=\frac{\phi(x)+\phi(y)}{2}.$$

Without knowing of my result, Shigeru Haruki took up the subject and in 1979 [**4**] went somewhat further. He proved that the more general equation

$$\frac{f(x)-g(y)}{x-y}=h(x+y) \qquad (x\neq y;\, x,y\in\mathbb{R}), \tag{3}$$

containing three unknown functions, also has $f(x)=g(x)=ax^2+bx+c$ and $h(x)=ax+b$ as its only solutions, again without any regularity conditions. Also his proof went by reduction to the Jensen equation.

I was reminded of my 'forgotten' result by P. Volkmann's talk [**6**] at a meeting in August 1983. During the same meeting I found an even simpler way of solving (2), which applies also to (3) and for which the Jensen equation is not needed. This proof, which I want to share with you here, remains valid for fields of characteristic different from 2. You may wish to stay first with real functions of real variables and check later that our proof holds also on the more general structure.

Let us assume that f, g, h are defined for all elements of our field (in particular, for all real numbers, if you so prefer). The equation

$$\frac{f(x)-g(y)}{x-y}=h(x+y) \qquad (x\neq y) \tag{3'}$$

can be written as

$$f(x)-g(y)=(x-y)h(x+y), \tag{4}$$

which shows no trace of derivatives or mean values. Unlike (3′), this equation (4) would make sense also if $x=y$ were permitted, which would give an easy proof that $g(x)=f(x)$, but we can proceed further under the restriction $x\neq y$. By interchanging x and y in (3′) (or (4)), we obtain

$$f(x)-g(y)=g(x)-f(y),$$

so

$$f(x)-g(x)=g(y)-f(y).$$

Since the left-hand side depends only upon x and the right only upon y, both have to be constant, say c. So now $f(t)-g(t)=c$, but also $g(t)-f(t)=c$, thus $c=-c$, that is,

$$2c=0 \quad \text{and so} \quad c=0. \tag{5}$$

Therefore

$$g(x)=f(x) \qquad \text{for all } x. \tag{6}$$

Consequently, equation (4) becomes

$$f(x)-f(y)=(x-y)h(x+y) \tag{7}$$

which is also true for $x=y$. Of course, if f satisfies (7), so does $f+b$ (b is constant). Therefore we may suppose without loss of generality that $f(0)=0$. Put $y=0$ into (7) in order to get

$$f(x)=xh(x). \tag{8}$$

This equation may be used to transform (7) into

$$xh(x)-yh(y)=(x-y)h(x+y). \tag{9}$$

Again, if this is satisfied by h, it is satisfied also by $h+c$, so we may suppose that $h(0)=0$. Therefore, puting $x=-y$ into (9), we get

$$-yh(-y)=yh(y);$$

that is, h is an odd function. We take this into consideration when replacing y by $-y$ in (9), getting

$$xh(x)-yh(y)=(x+y)h(x-y).$$

Comparison with (9) gives $(x-y)h(x+y)=(x+y)h(x-y)$ and substituting

$$u=x+y, \quad v=x-y \tag{10}$$

produces the equation

$$vh(u)=uh(v)$$

for all u,v, thus (choosing $v=v_0$, a nonzero constant, and $a=h(v_0)/v_0$)

$$h(u)=au.$$

If we do not assume $h(0)=0$, we have in general

$$h(u)=au+b.$$

By (8) this gives $f(x)=x(ax+b)$ and, if we do not assume $f(0)=0$, but take (6) into consideration, then

$$f(x)=g(x)=ax^2+bx+c.$$

So we have indeed proved that all solutions of (3′) (and also of (4)) are of the form

$$f(x)=g(x)=ax^2+bx+c, \quad h(x)=ax+b, \tag{11}$$

where a,b,c are arbitrary constants, as asserted. Straightforward substitution shows that all functions of the form (11) satisfy (3′) and (4).

If you considered throughout the proof the variables to be real numbers, you may go through it again, replacing real numbers by elements of an arbitrary field. You will notice that everything works, except that in two steps we need that the characteristic of the field be different from 2. The first is in (5) where $2c=0$ was supposed to imply $c=0$ which is true only if the characteristic is different from 2. The other is the supposition that the system of equations (10) has solutions x,y for arbitrarily given u,v. These ($x=(u+v)/2$, $y=(u-v)/2$) again exist only if the characteristic is different from 2. But that is all that needs to be supposed. So we have proved the following.

THEOREM. *The general solutions of* (3′) *in a field of characteristic different from* 2 *are given by* (11), *where a,b,c are arbitrary constants in the field. The solutions are the same for equation* (4) *in which $x=y$ is allowed.*

For fields of characteristic 2 the theorem is not true. To see this, suppose that our variables move in the field $\mathbb{Z}_2$, the integers modulo 2. Then the functions $f(x)=x$, $g(x)=x+1$, $h(x)=x+1$ satisfy (3′) and (4) for $x\neq y$ (but not for $x=y$), while $f(x)=g(x)=h(x)=x$

satisfy (3′) (for $x \neq y$) and (4) (also for $x = y$), but neither is of the form (11). (The first example is due to Ulrich Daepp.)

REMARKS. There are several interesting generalizations of (1), which do involve both means and derivatives.

One question is: *what kind of mean values* (other than arithmetic means) *can appear on the right-hand side of equations like* (1)? In other words, for what mean values $M(x, y)$ do there exist differentiable functions f such that

$$\frac{f(x) - f(y)}{x - y} = f'[M(x, y)]. \tag{12}$$

R. Bojanić [**3**] has found necessary and sufficient conditions in the form of partial differential equations for M.

Another question arises in considering **quasiarithmetic means**, that is, those which can be represented in the form

$$M(x, y) = F^{-1}\left(\frac{F(x) + F(y)}{2}\right),$$

where F is continuous and strictly monotonic. (For the arithmetic mean, $F(x) = x$.) *When can* M *in* (12) *be a quasiarithmetic mean*? Geometrically, this question asks for what curves will the abscissa of the point where the tangent is parallel to a chord be a quasiarithmetic mean of the abscissas of the endpoints of the chord for every chord. G. Aumann proved [**2**], (see also [**3**]), that arcs of conic sections are exactly the curves with this property (on an interval) even if (12) is replaced by

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'[M(x, y)]}{g'[M(x, y)]}.$$

(We talk about *arcs* of conic sections because, for instance, a complete ellipse is not the graph of a *function*, since it is not single valued.) This is a nice generalization of the property of parabolas which served as our point of departure.

References

[1] J. Aczél, Bemerkungen 11–12, Tagungsbericht Funktionalgleichungen 7–11 Okt. 1963 (2. Tagung), Mathematisches Forschungsinstitut Oberwolfach, 1963, p. 14.

[2] G. Aumann, Über einen Zusammenhang zwischen Mittelwerten, Streckenrechnung und Kegelschnitten, Tōhoku Math. J., 42 (1936) 32–37.

[3] R. Bojanić, Sur la formule des acroissements finis, Acad. Serbe Sci. Publ. Inst. Math., 3 (1950) 219–226.

[4] Sh. Haruki, A property of quadratic polynomials, Amer. Math. Monthly, 86 (1979) 577–579.

[5] T. L. Saaty, Modern Nonlinear Equations, McGraw-Hill, New York, 1967.

[6] P. Volkmann, Eine Funktionalgleichung für dividierte Differenzen. The Twenty-first International Symposium on Functional Equations, August 6–13, Konolfingen, Switzerland, Aequationes Math., 26 (1984) 225–294, in part. pp. 252–253.

0 For A Solution!

The solution to my problem I sought,
When I hit on a brilliant thought.
By a method involved
The equation was solved
And the answer was: nought equals nought.

—MARTA SVED