

A Surface With One Local Minimum

J. MARSHALL ASH

*DePaul University
Chicago, IL 60614*

HARLAN SEXTON

*NO SC Code 632B
San Diego, CA 92152*

Consider the following statement.

STATEMENT F. *A smooth surface $(x, y, f(x, y))$ with one critical point which is a local, but not a global, minimum must have a second critical point.*

Here smooth may be taken to mean that f is infinitely differentiable. A point (a, b) is a critical point if

$$\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0.$$

When asked what we thought of this statement, our first reaction was that it ought to be true since a one-dimensional analogue is. After some thought we came up with an “almost rigorous” argument that supported statement F. Somewhat later, assisted by a geometric idea of P. Ash of St. Joseph’s University (the first author’s brother), we found a counterexample to statement F. Finally, we applied an important principal of mathematical research which A. Zygmund of the University of Chicago has frequently expounded in his seminar: Never be stopped by a counterexample; instead find out what is really happening.

Guided by this maxim, we were able to add a small hypothesis (suggested by William Browder of Princeton University) which did force the conclusion of statement F to follow. More explicitly, we have

THEOREM T. *Let $f: R^2 \rightarrow R$ be continuously differentiable and have a local, nonglobal minimum. If, further, f is proper ($f^{-1}(K)$ is compact whenever K is a compact subset of the range) then f must have at least one additional critical point.*

The theorem appears to be unpublished folklore, and we will supply a proof after first presenting some thoughts about statement F, then a counterexample.

In considering statement F, an obvious question to ask is: What happens in one dimension? Let $f: R \rightarrow R$ be smooth and have a local minimum at 0, for example, $f(0) = 0$ and $f(x) > 0$ for all $|x| < \delta$. If 0 is not a global minimum then we must have $f(a) < 0$ for some a , say $a > 0$. But then $f(\delta/2) > 0$, $f(a) < 0$ and the intermediate value theorem gives $f(b) = 0$ for some b in $(\delta/2, a)$. But $f(0) = f(b) = 0$ so Rolle’s Theorem implies the existence of $c \in (0, b)$ with $f'(c) = 0$. Thus f has a second critical point at c .

Encouraged by this evidence for statement F, let us perform a thought experiment. Suppose we pour water onto the surface from a spout located directly above the critical point. The water will steadily rise. Evidently it must overflow sooner or later if the local minimum is not absolute. The point at which the overflow first occurs must be a second critical point. This argument has strong intuitive appeal, but the idea that the “bowl” around the local minimum has finite volume is implicitly incorporated into the assumption of eventual overflow.

The following counterexample shows that statement F is false; although discovered independently, it is somewhat similar to one given by David Smith [1, p. 750]. Define

$$f(x, y) = \frac{-1}{1+x^2} + (2y^2 - y^4) \left(e^x + \frac{1}{1+x^2} \right). \quad (1)$$

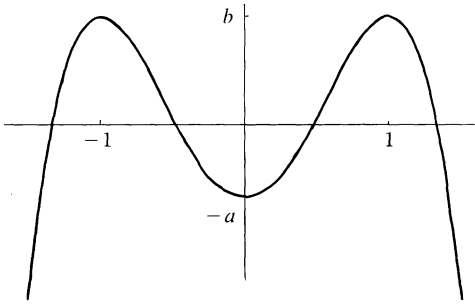


FIGURE 1. Section $x = \text{constant}$.

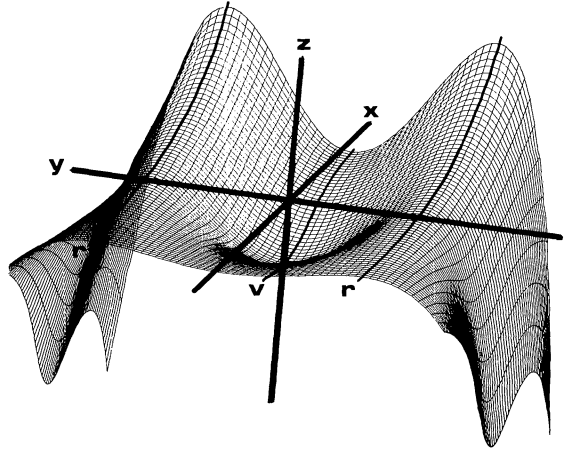


FIGURE 2

The function f is as differentiable as you like (in fact, it is real-analytic). The point $(0, 0)$ is a local, not global minimum, and there are no other critical points.

To find any critical point(s) of f , first consider sections of the form $x = \text{constant}$. Then $f = f(y) = -a + (2y^2 - y^4)(b + a)$ has critical points at $y = 0, 1$, and -1 . Now consider sections of the form $y = \text{constant}$. As FIGURE 1 shows, we need only look at the sections $y = 1, y = 0$, and $y = -1$, since nowhere else is $\partial f / \partial y = 0$. On the sections $y = -1, y = 1, f = e^x$ so $\partial f / \partial x = e^x$ is always positive. On the section $y = 0, f(x) = -1 / (1 + x^2)$, so $\partial f / \partial x = 2x / (1 + x^2)^2 = 0$ only at $x = 0$. Thus $(0, 0)$ is the only critical point.

Since $f(0, 0) = -1 > -17 = f(0, 2)$, the point $(0, 0)$ is not a global minimum. It only remains to show that $(0, 0)$ is indeed a local minimum. We have

$$\begin{aligned}
 f(x, y) &= \left[\frac{-1}{1+x^2} \right] + (2y^2 - y^4) \left(e^x + \frac{1}{1+x^2} \right) \\
 &= \left[-1 + \frac{x^2}{1+x^2} \right] + y^2(2-y^2) \left(e^x + \frac{1}{1+x^2} \right) \\
 &= f(0, 0) + \left\{ \frac{x^2}{1+x^2} + y^2(2-y^2) \left(e^x + \frac{1}{1+x^2} \right) \right\}. \tag{2}
 \end{aligned}$$

If (x, y) is in the unit disc about $(0, 0)$, then the quantity in curly brackets in (2) is positive except when $x = y = 0$. This shows $(0, 0)$ to be a local minimum.

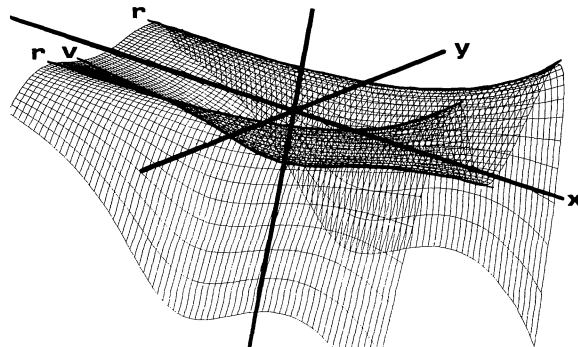


FIGURE 3

We restate the above proof geometrically, using FIGURES 2 and 3. (The 3 coordinate axes are separately scaled here to bring out the salient features.) Each section “parallel” to the y -axis has the shape shown in FIGURE 1 with one dimensional critical points occurring on the ridges labeled r and on the valley bottom labeled v . Since the ridges grow like e^x the tangent plane can be horizontal only at some point of v . The only such point of v is the place where the z axis pierces v , which is the local minimum.

Proof of Theorem T

Assume f satisfies the hypotheses of the theorem. Let f have its local minimum at $A \in R^2$ and let $B \in R^2$ be such that $f(B) < f(A)$. Pour water onto the surface determined by f , above the point A . Then either (i) the water level will rise to arbitrarily great heights, or (ii) the water level will asymptotically approach a finite height h (as actually happens for f defined by (1)), or (iii) the water will overflow. We will proceed to eliminate cases (i) and (ii), in which case our earlier argument will become the proof of the theorem.

In R^2 let \overline{AB} be the line segment joining A and B . Then the continuous function f attains a maximum, say m , on the compact set \overline{AB} . The water cannot rise to a height greater than m without spilling so that case (i) is impossible.

If case (ii) were to occur, then the water would be held by a reservoir which would lie over a subset S of the compact set $f^{-1}([f(A), h])$ and whose depth would be everywhere less than $h - f(A)$. Such a reservoir would have finite volume (less than the product of the measure of S with $h - f(A)$). Since we may pour as much water as we like, case (ii) is impossible.

References

[1] Philip Gillett, *Calculus and Analytic Geometry*, 2nd ed., D. C. Heath, Lexington, Mass., 1984.

“The Only Critical Point in Town” Test

IRA ROSENHOLTZ

LOWELL SMYLIE

University of Wyoming

Laramie, WY 82071

When searching for absolute extrema of functions of a single variable, it is often convenient to apply the well-known “Only Critical Point in Town” Test: *If f is a continuous function on an interval, which has a local extremum at x_0 , and x_0 is the only critical point of f , then f attains an absolute extremum at x_0 .* A natural question which arises is “Is the corresponding statement true for functions of two variables (say defined over the entire plane)?” Since our colleagues were evenly split on the question (both halves being quite adamant), and neither a proof nor a counterexample was readily available, we set to work trying to find one.

Progress came slowly at first. Then one bright Monday morning we exchanged pictures of what we thought the level curves of a counterexample might look like. And believe it or not, we both had the same picture!—right down to the location of the mountain, the river bed, and the cliff! It looked something like FIGURE 1. Most of our colleagues were convinced by our picture, but a few remained rightfully skeptical: They wanted a formula. We did too.

Then we noticed something—our level curve picture seemed to have a “saddle point at infinity.” So we tried to mimic this.