Applying the exponential function and taking the limit for $n \to \infty$ leads to

$$\left(1+\frac{n}{n}\right)\left(1+\frac{n-1}{n}\right)\cdots\left(1+\frac{1}{n}\right) \sim e^{n\cdot(2\ln 2-1)+\frac{\ln 2}{2}}, \quad n \to \infty.$$

If we multiply this result by n^n and rewrite the right hand side, we get the following asymptotic estimate for (5):

$$\frac{(2n)!}{n!} \sim \frac{2^{2n} n^n \sqrt{2}}{e^n}, \quad n \to \infty. \tag{7}$$

By multiplying the estimates in (4) and (7), we get Stirling's formula (1).

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Summary In this note an elementary proof of Stirling's asymptotic formula for $n!$ is given. The proof uses the Wallis formula for π and the trapezoidal rule for the calculation of a definite integral, with error estimate.

A Note on Disjoint Covering Systems—Variations on a 2002 AIME Problem

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We will consider two questions about covering systems, both suggested by Problem 9 from the 2002 American Invitational Mathematics Examination (AIME) [**1**]. The AIME problem, in effect, is the following:

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AMIE PROBLEM. Harold, Tanya, and Ulysses paint a very long fence. Harold starts with the first picket and paints every hth picket; Tanya starts with the second picket and paints every tth picket; and Ulysses starts with the third picket and paints every uth picket. If every picket gets painted exactly once, find all possible triples (h, t, u).

After some analysis the only such triples can be seen to be $(3, 3, 3)$ and $(4, 2, 4)$. Hint: if (h, t, u) is such a triple then $h^{-1} + t^{-1} + u^{-1} = 1$.

We use this problem to introduce the topic of covering systems. Our work on this subject led us to the following two questions that are inspired by Problem 9, and for which we will use the same metaphor.

QUESTION 1. Six students, Wesley, Lindsay, Ryan, Nicole, Jared and Kelly are painting a very long fence. Wesley paints every wth picket, Lindsay every lth picket, Ryan every rth picket, Nicole every nth picket, Jared every jth picket and Kelly every kth picket. While painting, they notice that pickets $11, 12, \ldots, 34$ are all painted exactly once. When they are finished the entire fence is painted. Did they paint each picket exactly once?

QUESTION 2. The six students paint as before. After a while, they notice that each of the pickets starting from the 9th picket to the 72nd picket is painted exactly once. If they continue in this fashion, will they paint each picket? If so, will they paint each picket exactly once?

(To be clear, we mean for each student to paint an entire congruence class. For example, Wesley may start at any picket, but then must paint every wth picket in each direction.)

We will answer these questions affirmatively. We start by introducing basic definitions and useful known facts. As is customary for integers a and $m > 0$, by the congruence class $a \pmod{m}$, we mean the set $\{a + jm : j \in \mathbb{Z}\}$.

We need some definitions. Let $\mathcal{A} = \{x \equiv a_i \pmod{m_i} : 1 \le i \le k\}$ be a system of congruences. Then

- $\mathcal{A}$ is called a *covering system* (CS) if every integer x satisfies at least one of the congruences in $\mathcal{A}$.
- If $\mathcal{A}$ is a CS and deletion of any of the congruences causes $\mathcal{A}$ to no longer be covering then $\mathcal{A}$ is said to be a *regular covering system* (RCS). Some authors call this type of covering system *irredundant.*
- If every integer satisfies exactly one congruence in $\mathcal{A}$ then it is said to be a *disjoint covering system* (DCS). This is also referred to as an *exact covering system.*

Let $c(x)$ be the number of congruences that $x \in \mathbb{Z}$ satisfies. This function is called the *covering function.* If a system is covering then $c(x) \ge 1$ for all $x \in \mathbb{Z}$, and if a system is a DCS, then $c(x) = 1$ for all $x \in \mathbb{Z}$.

We introduce two lemmas which will prove to be useful throughout this paper. Both were proved by R. J. Simpson [**6**]:

LEMMA 1. (SIMPSON'S LEMMA) *If* $\{x \equiv a_i \pmod{m_i} : 1 \le i \le k\}$ *is an RCS, and a prime* p *divides some modulus, then the set* $\{a_i : p \mid m_i\}$ *contains a full set of residues modulo* p.

LEMMA 2. (SIMPSON'S INEQUALITY) *If* $\{x \equiv a_i \pmod{m_i} : 1 \le i \le k\}$ *is an RCS and* $\prod_{i=1}^{t} p_i^{\alpha_i}$ *is the canonical prime factorization of* $\operatorname{lcm}(m_1, \ldots, m_k)$*, then* $k - 1 \ge \sum_{i=1}^{t} \alpha_i (p_i - 1)$.

(Simpson's inequality is corollary 2 of [**6**].)

The affirmative answer to Question 1 follows from the following theorem. Observe that six students are told that the entire fence is painted, and they painted $24 = 3 \cdot 2^{6-3}$ consecutive pickets exactly once.

THEOREM 1. *If $\mathcal{A} = \{x \equiv a_i \pmod{m_i} : 1 \le i \le k\}$ is an RCS such that $c(x) = 1$ for* $\max\left\{2, 3 \cdot 2^{k-3}\right\}$ *consecutive integers, then $\mathcal{A}$ is a DCS.*

Proof. We proceed by induction on k, the number of congruences in the system. If $k = 2$ then an RCS must have the form $\{x \equiv 0 \pmod 2, x \equiv 1 \pmod 2\}$. So $c(0) = 1 = c(1)$, thus it is a DCS satisfying the theorem. If $k = 3$, there are only two RCS (up to shifting): $\{x \equiv 0 \pmod 3, x \equiv 1 \pmod 3, x \equiv 2 \pmod 3\}$ and $\{x \equiv 0 \pmod 2, x \equiv 1 \pmod 4, x \equiv 3 \pmod 4\}$. Both are DCS which satisfy the theorem.

Now let $k_0 \ge 4$ and assume that the theorem holds for all $k < k_0$. Suppose, for the sake of contradiction, that there exists an RCS, $\mathcal{A} = \{a_i \pmod{m_i} : 1 \le i \le k_0\}$, which has the property that $c(x) = 1$ for $3 \cdot 2^{k_0-3}$ consecutive integers but which is not a DCS.

Clearly there is some integer x for which $c(x) > 1$. By shifting the system, we can assume without loss of generality that $c(0) > 1$ and $c(1) = c(2) = \cdots = c(3 \cdot 2^{k_0-3}) = 1$.

We shall prove that 0 cannot be covered by two congruences with distinct prime moduli. Suppose that $\mathcal{A}$ contains the congruences $x \equiv 0 \pmod p$ and $x \equiv 0 \pmod q$ where p and q are distinct primes. Clearly $c(pq) \ge 2$. If $pq \le 3 \cdot 2^{k_0-3}$ then this contradicts the fact that $c(x) = 1$ for $1 \le x \le 3 \cdot 2^{k_0-3}$. Let us assume that $pq > 3 \cdot 2^{k_0-3}$. By Simpson's inequality we see that $k_0 \ge p + q - 1$ which implies $pq > 3 \cdot 2^{p+q-4}$ which is equivalent to $16/3 > 2^p/p \cdot 2^q/q$. The function $2^x/x$ is increasing for $x \ge 2$ so the right-hand product has its smallest value, $16/3$ when $p = 2$ and $q = 3$ which contradicts the assumption that $pq > 3 \cdot 2^{k_0-3}$.

It follows that there is at least one congruence covering 0 that involves a modulus, say m, that is composite. Let p be a prime divisor of m. By Simpson's Lemma $p \le k_0$.

We now partition $\mathcal{A}$ into three collections:

$$\mathcal{A}_1 = \{x \equiv a_i \pmod{m_i} \in \mathcal{A} : p \nmid m_i\}$$
$$\mathcal{A}_2 = \{x \equiv a_i \pmod{m_i} \in \mathcal{A} : p \mid m_i \text{ and } p \mid a_i\}$$
$$\mathcal{A}_3 = \{x \equiv a_i \pmod{m_i} \in \mathcal{A} : p \mid m_i \text{ and } p \nmid a_i\}.$$

By Simpson's lemma, $|\mathcal{A}_3| \ge p - 1$. Thus, $|\mathcal{A}_1 \cup \mathcal{A}_2| \le k_0 - (p-1)$. Now, let us form a new collection $\mathcal{A}^* = \mathcal{A}_1^* \cup \mathcal{A}_2^*$ where

$$\mathcal{A}_1^* = \{x \equiv a_i p^{-1} \pmod{m_i} : x \equiv a_i \pmod{m_i} \in \mathcal{A}_1\}$$
$$\mathcal{A}_2^* = \{x \equiv a_i/p \pmod{m_i/p} : x \equiv a_i \pmod{m_i} \in \mathcal{A}_2\}$$

and p^{-1} is the multiplicative inverse of p modulo m_i.

Consider integers pn, where n is any integer. Each of these must belong to at least one congruence in $\mathcal{A}_1 \cup \mathcal{A}_2$ so n belongs to the corresponding congruence in $\mathcal{A}^*$. In particular, 0 belongs to at least two congruences and $1, 2, \ldots, \lfloor 3 \cdot 2^{k_0-3}/p \rfloor$ each belong to exactly one. It is easily checked that $\lfloor 3 \cdot 2^{k_0-3}/p \rfloor \ge \max\{2, 3 \cdot 2^{k_0-3-(p-1)}\}$ which means that $\mathcal{A}^*$ is a counterexample to the inductive hypothesis. The theorem follows. ∎

The bound in Theorem 1 cannot be improved, as the following example demonstrates. For $k \ge 4$, consider the RCS

$$x \equiv \begin{cases} 2^{j-1} & \pmod{2^j}, \text{ for } 1 \le j \le k-3 \\ 2^{k-3} & \pmod{3 \cdot 2^{k-3}} \\ 2^{k-2} & \pmod{3 \cdot 2^{k-3}} \\ 0 & \pmod{3 \cdot 2^{k-4}} \end{cases}$$

In this system $c(x) = 2$ where $x = 3 \cdot 2^{k-4} + n \cdot 3 \cdot 2^{k-3}$ for $n \in \mathbb{Z}$, and $c(x) = 1$ for other integers. So $c(x) = 1$ for $3 \cdot 2^{k-3} - 1$ consecutive integers, but the system is not disjoint.

To answer the second question, we must appeal to another fact about covering systems. In 1962 Erdős [**3**] proved that a system of congruences will cover every integer if it covers all integers in the interval $[1, k2^k]$ where k is the number of congruences. This was not improved until 1970 when R. B. Crittenden and C. L. Vanden Eynden in [**2**] proved the following:

LEMMA 3. *Let $a_1, a_2, \ldots, a_n, m_1, m_2, \ldots, m_k$ be given, with m's positive. Suppose there exists an integer x_0 satisfying none of the congruences*

$$x \equiv a_i \pmod{m_i}, \quad i = 1, 2, \ldots, k$$

Then there is such an x_0 among $1, 2, \ldots, 2^k$.

A shorter sequence will not suffice in Lemma 3, as the following example shows

$$\{x \equiv 2^{j-1} \pmod{2^j}, 1 \le j \le k\}$$

This system covers $2^k - 1$ consecutive integers exactly once, but 0 is not covered by any congruence so the system is not covering. Now the answer to Question 2 is provided by the following Corollary:

COROLLARY 1. *If a system with k congruences covers 2^k consecutive integers exactly once, then it is a DCS.*

Proof. This follows immediately from Theorem 1 and the Theorem of Crittenden and Vanden Eynden. ■

Finally we provide an upper bound for the least common multiple of the moduli in an RCS.

THEOREM 2. *Given any regular covering system* $\{x \equiv a_i \pmod{m_i} : 1 \le i \le k\}$, *it follows that* $\operatorname{lcm}(m_1, \ldots, m_k) \le 2^{k-1}$.

Proof. Let $\operatorname{lcm}(m_1, \ldots, m_k) = \prod_{i=1}^t p_i^{\alpha_i}$ be the canonical prime factorization. We wish to maximize this value subject to Simpson's inequality [**6**, corollary 2], $k - 1 \ge \sum_{i=1}^t \alpha_i (p_i - 1)$. For every prime p we have $p \le 2^{p-1}$ therefore

$$\log_2 \prod_{i=1}^t p_i^{\alpha_i} = \sum_{i=1}^t \alpha_i \log_2 p_i \le \sum_{i=1}^t \alpha_i (p_i - 1) \le k - 1$$

by Simpson's inequality. The result follows. ■

The interested reader should refer to [**5**] and [**7**] for surveys of the work done on covering systems of congruences. Both sources also contain extensive bibliographies on the topic of covering systems.

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Summary A *covering system* is a system of k arithmetic progressions whose union includes all integers. It is a *disjoint covering system* (or *exact covering system*) if the progressions are also pairwise disjoint, so that each integer is covered exactly once. This paper presents upper bounds on the number of consecutive integers which need to be checked to determine whether a covering system is a disjoint covering system. The bounds depend only on the number of congruences in the system. The results provide an analog of a theorem by R. B. Crittenden and C. L. Vanden Eynden from 1969 and are presented as solutions to some variations of a 2002 AIME Problem about painting a picket fence.

Convexity and Center of Mass

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In differential calculus, we encounter the notion of convexity as one of the main characteristics of the graph of certain functions; and in integral calculus, centers of mass (or centroids) are discussed as an important practical application. In this note we explore a link between these two concepts—a link that appears to have gone unnoticed despite its simplicity and plausibility.

A function is called "convex" (on an interval) if its graph is "concave up." We show that centers of mass preserve convexity in the following sense:

THEOREM 1. *Suppose that the functions f and g are continuous and convex on an interval $[a, b]$ and satisfy $f(x) \geq g(x)$ for all x in $[a, b]$. Then, for any partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, the centroids of the regions bounded by the lines $x = x_i$ and by the graphs of f and g are the vertices of a convex polygonal curve.*

From elementary geometry, we know that the centroid of a trapezoid lies on the median that connects the midpoints of the two parallel sides. Trapezoids can be treated as a special case of Theorem 1 in which both f and g are linear. In that case the centroids of the regions all fall on the line $y = \frac{1}{2}(f(x) + g(x))$, and the polygonal curve formed by them can be viewed as both convex and concave.

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