A Professional Program for Preparing Future High School Mathematics Teachers

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Introduction. There has been growing attention in recent years to ensuring that every K-12 student is taught by a high-quality teacher. Additionally, the Common Core State Standards for Mathematics (CCSSM) place additional demands on teachers (see http://www.corestandards.org/Math/). This has led to a call for greater professionalism for teachers in admission standards, in their education, in their responsibilities and in their status [1, 2]. The recent Mathematical Education of Teachers II report (MET II) from the Conference Board of the Mathematical Sciences [3] has many recommendations to foster such professionalism among teachers of mathematics (see http://cbmsweb.org/MET2/index.htm). As in the first MET report, the high school chapter of the MET II report calls for a greater connection between the college mathematics taken by future teachers and the high school mathematics they will teach. The MET II recommendations, if widely adopted, will have a powerful impact on the future preparation of high school mathematics teachers.

The traditional mathematics major is a liberal arts education that is concerned foremost with the power of mathematical reasoning and theory. Its graduates are prepared for a range of careers and graduate study. On the other hand, teacher education is focused on preparation for a teaching career. Mathematics for teaching has been described as "the body of mathematics that is important for teachers to know in order to be able to successfully manage the mathematical demands of their professional practice, i.e., teaching mathematics to children" [4]. Hyman Bass, Zalman Usiskin, and others call mathematics for teaching "applied mathematics"; H. H. Wu calls it "mathematical engineering" [4]. For this reason, this document is written from the point of view that a traditional liberal arts major in mathematics is neither necessary nor sufficient preparation for teaching high school mathematics (unless it is followed by graduate training in mathematics education).

This report presents recommendations for a professional program for preparing high school mathematics teachers that builds on the MET II report. The program includes a re-design of some courses and additional courses in the related areas. Such curricular changes are likely to take several years. To offer the re-designed courses along with existing courses, mathematics departments with small numbers of prospective high school teachers may need to consider regional consortia, distance learning, or co-convening courses with in-service teachers seeking graduate credit or professional development. This report also discusses ways to maximize the value of a less extensive program. Regardless of what is feasible for a particular department, the design criteria for this program and recommendations for individual courses should be useful to all departments.

Many mathematics faculty may think that high school mathematics is neither hard to learn nor very deep. However, the reality is that it took thousands of years, informed by the work of many of history's greatest mathematical minds, to discover and refine the content of today's high school mathematics and to frame it in a fashion that lays the foundation for collegiate mathematics. Thus, it is a substantial intellectual accomplishment, fully worthy of a type of mathematics major, to understand this mathematics and its extensions deeply in order to build a framework for excellent high school instruction.

Evolving Goals. The goals of a mathematics teacher's pre-service education should reflect the goals of K-12 mathematics education. From the turn of the 20th century until about 60 years ago, the primary mathematical goal of school education was mastery of the four operations of arithmetic, for the integers and fractions, along with applications of this arithmetic to commercial problems. In 1950 only about 25% of high school students took a course in algebra [13]. Historically, high school mathematics teachers' collegiate education focused on the in-depth study of high school mathematics including advanced topics such as spherical trigonometry [6]. Teachers' colleges did not offer calculus until the mid-20th century. With high schools today offering college-level calculus and statistics, teachers' content knowledge has increased. There is an appreciation that future high school mathematics teachers need courses in analysis, algebra, and geometry that examine foundational issues in school mathematics from an advanced perspective.

There are two basic goals today of K-12 mathematics for students: (i) to acquire mathematical skills and knowledge needed to understand quantitative topics and models that students will see in subsequent courses, careers and daily lives; and (ii) to develop quantitative reasoning and problem-solving skills. This latter goal has long been valued for developing skill at analyzing problems in non-quantitative settings. There is a third (often unstated) purpose that builds on (i) and (ii): (iii) to develop the capacity to reconstruct, apply and extend one's mathematical knowledge and reasoning throughout one's career. Implicit in (iii) is an appreciation of the power of mathematical reasoning and of the structure underlying mathematical knowledge. The education of future high school mathematics teachers should prepare them to offer instruction that addresses these three goals, through mastery of the mathematics and the methods to teach it effectively.

Two false dichotomies. Two false dichotomies should be avoided when discussing school mathematics and how teachers are prepared to teach it. The first is that mathematics is either practical or abstract. While much of mathematics was developed for very practical reasons, its learning and use are greatly simplified by higher-order mathematical practices—abstraction, generalization, theories, and patterns of reasoning. These practices may seem far removed from practical considerations, but they should not be treated that way. Teaching students to employ theory and reasoning skills to learn, retain, and apply high school mathematics is the core challenge for high school teachers. Reasoning skills such as decomposing problems into parts, working simplified problems, and conjecturing have universal applicability. See the MET II report [3, pp. 55-56] for an extensive discussion of the role of reasoning skills in the education of future mathematics teachers.

The second dichotomy involves the perceived disconnect between the content of school mathematics and collegiate mathematics. It was long assumed by mathematics faculty that a standard mathematics major, slightly modified for teachers, would be an adequate preparation for teaching high school mathematics, even though most upper-division mathematics courses do not often make explicit connections to high school mathematics. That belief has been questioned and studies have supported this skepticism [3, p.11; 5, 8, 9].

This report supports the MET II's call to connect mathematics major courses for future mathematics teachers more directly to high school mathematics**.** The core of this connection is three courses in high school mathematics from an advanced perspective. We recommend that all collegiate mathematics instruction, not just for future teachers, address the first dichotomy by emphasizing the interplay between individual problems with general theory—problems motivate theory and theory simplifies and organizes the analysis of problems—and between specialized problem-solving techniques and more universal patterns of analytic reasoning. Future teachers need to see this interplay constantly to ensure that they embed it in all their teaching. This treatment will strengthen their students' preparation for further study in mathematics or for careers in mathematical fields. Further, the professional education of teachers requires that the cognitive goals put forth in the Overview to this Guide be fully addressed.

A professional program. This professional program for educating high school mathematics teachers, like other professional programs, has a curriculum focused on the preparation needed to be a competent practitioner upon graduation. For mathematics departments that prepare few high school mathematics teachers or where other factors make or primary recommendations impractical, we offer suggestions for various ways to accommodate the spirit of the primary recommendations with fewer changes along with a capstone course for teachers to connect their major courses with high school mathematics.

The proposed curriculum

Many parts of the following curriculum duplicate recommendations in the MET II report's high school chapter (freely available at http://cbmsweb.org/MET2/index.htm); the reader is referred to that chapter for the rationale for those courses for teachers and for examples of how these courses can give deeper understanding about topics in high school mathematics from an advanced viewpoint. Readers should also refer to discussions of most of these courses in other sections of this Guide. Comments here will concentrate on new courses for future teachers or new approaches to standard courses.

Note: In states requiring a master's degree for teaching licensure, parts of this program could be incorporated in the graduate studies.

A. Lower division mathematics courses

We recommend lower division preparation that includes the study of single and multivariable calculus, differential equations, linear algebra, and transition to proof. We also recommend two courses in statistics and data analysis. Comments about these lower division courses follow.

Calculus has so many constituencies that special courses for teachers are not realistic; see the Calculus subgroup report. However, as recommended in the MET II report, sections of a course can be dedicated to prospective teachers. Similarly, see the Differential Equations subgroup report for suggestions for an appropriate DE course. As recommended in that subgroup report, it is important to use technology and introduce modeling extensively in this course.

Matrix-oriented linear algebra is mostly linear analysis. Vectors and matrices arise in a range of problems in analysis, and in a variety of important applicable models, such as Markov chains. Matrices have a growing role in high school mathematics. The linear algebra course for teachers should start with the basics of analytic geometry in 3-space (e.g., equations of lines and planes) and progress to address matrix-oriented linear algebra theory, a powerful and useful, yet accessible, theory in mathematics. Linear algebra epitomizes the interplay between theory and concrete problems discussed above. See the Linear Algebra section of this Guide for details about the suggested matrix-oriented course. It is also appropriate for mathematics teacher preparation to blend a matrix-oriented approach with a more abstract approach, if the more abstract approach is necessary for the non-teaching mathematics majors who also take the course. Also see the linear algebra discussion in the MET II report 3; p. 57 - 58].

A Transition to Proof course should be centered on an important mathematical subject whose problems are used to develop proof skills; for example, number theory or discrete structures. Future teachers should always see that proofs have a purpose, and that proofs can build theories that organize and explain mathematical topics. The text by Rotman [11] is based on topics in high school geometry and algebra, making it appropriate as part of a high school teacher preparation program. See the Transition to Proof section of this Guide for further suggestions.

The Common Core standards call for a major increase in statistics instruction in the high schools, reflecting the pervasive role of statistics in business, science, and personal decisions [12; p. 79-83]. We recommend two statistics courses as part of the mathematics teaching major, although with a very different focus from a calculus-based probability-statistics course most future high school mathematics have historically taken. The justification for two courses is that, because so little statistics appeared in the pre-Common Core high school curriculum, future teachers need to learn the Common Core statistics material well in the first course, and then take a second college statistics course that builds on the Common Core material. Even as students are educated in high schools using the Common Core standards in statistics, statistical inference is so important in modern society that it is easy to argue that teachers' investment in two courses in statistics is justified. Two course syllabi for this second course suggested by statistics educators are in the statistics section of this Guide, and further information will be available in the forthcoming *Statistical Education of Teachers*. The MET II high school chapter has recommendations for a single lower-division course for future teachers that can also serve as a higher-level introductory statistics course for other majors.

B. Upper-division mathematics courses

Our recommendations for upper-division courses have a core of coursework in abstract algebra, geometry, mathematical structures, and analysis, along with courses in most of the following topics: discrete mathematics, mathematical modeling, number theory, probability and history of mathematics. Our main emphasis and recommendation is that three courses address school mathematics content from an advanced viewpoint, as is recommended in the MET II report. We provide suggestions for how to organize mathematical content within three such courses. Regardless of the titles of the courses or how the mathematical content is organized within courses, our main recommendation for teacher preparation is that the program contain three courses with a purposeful focus on the mathematical content as it connects to high school mathematics.

The program in this Guide does not recommend separate "capstone" courses for teachers, although in the modifications section, below, we suggest an alternative pathway that would make use of a capstone for departments without flexibility to institute this report's full recommendation. MET II lays out how some of the material in these three courses could be developed, and we provide further details here. Future teachers often experienced 12 years of school mathematics, what is sometimes referred to as 'textbook mathematics' that was mechanical and lacking in understanding or mathematical precision. These courses for teachers will de-program teachers' faulty view of mathematics and rebuild knowledge of high school mathematics on a sound logical foundation. These courses should include a focus on how high school mathematics relates to the elementary and middle school math that comes before it and to the university mathematics that follows it.

A suggested syllabus for a *geometry* course would have two parts. The first (longer) part would cover Euclidean geometry with primary focus on results about triangles; characterizations of Pythagorean triples would also be discussed. These ideas lead to the Laws of Sines and Cosines and trigonometry from a geometric viewpoint. This part could also include the Greek way of showing that the areas of inscribed and circumscribed regular polygons approach the area of a circle. Triangle congruence would be understood from the point of view of rigid motion transformations and similarity from the point of view of rigid motion and dilations. Understanding geometry from the point of view of transformations is essential for teachers in addressing the geometry content specified by the Common Core State Standards. The second part would use analytic geometry, including conic sections, to re-prove earlier results in Euclidean geometry. It is appropriate for this course to use dynamic geometry software.

A suggested syllabus for a *mathematical structures* course would have two parts. The first (shorter) part would take a rigorous look at the rational number system, including place value arithmetic, the unique form of reduced fractions, and the fundamental theorem of arithmetic. The second part would look at the geometry of the plane from a structural point of view. One approach would be in terms of transformations; the other approach would be in terms of axiomatic systems, briefly surveying some of the other axiomatizations of the plane besides Euclidean geometry and including different formulations of parallelism and perpendicularity.

A suggested syllabus for an *analysis for teachers* course would have three parts. The first would start with rigorous proofs of results about real-valued functions, which lay the foundation for high school level material for manipulating polynomials and algebraic equations; then the extensions of the technique for completing the square and the quadratic formula. This leads to complex numbers and DeMoivre's theorem. The next part would look at the exponential and logarithm functions along with trigonometric and inverse trigonometric functions. The final part would study basic limits, including the decimal expansion of a number, and basic theorems about continuous functions. The latter would lead to derivatives, Riemann integrals, and the fundamental theorem of calculus.

In addition to the high school algebra topics in the mathematical structures course, we recommend an Abstract Algebra course focusing on algebraic structures encountered in high school mathematics. The content of an appropriate algebra course is found in the Algebra E course of the Abstract Algebra CASG in this Guide. Readers should also consult the comments about the modern algebra in the MET II report [3, pp. 59-60].

Likewise, see the appropriate sections of this Guide for recommendations about courses in discrete mathematics, mathematical modeling, number theory, probability and history of mathematics.

Practical adjustments of these new courses in high school mathematics from an advanced perspective. The courses for teachers presented here (other than, perhaps, a geometry course) are currently offered in only a few mathematics departments; UC-Berkeley and UC-Santa Barbara are the best known examples (for information on these courses, see www.ams.sunysb.edu/~tucker/BerkeleyMathEd.pdf and www.ams.sunysb.edu/~tucker/UCSBMathEd.docx). Individual courses and course sequences in high school mathematics from an advanced are becoming common; Montana State University, University of Nebraska, and the University of Arizona, for example, offer such course sequences. While we support the MET II recommendation that three such courses are needed for a sound mathematical preparation for future high school mathematics teachers, we offer suggestions for near-term strategies to cover much of the recommended content in geometry, structures, and analysis by modifying existing courses found in many departments.

The geometry material can be incorporated in the geometry course that most departments offer for future high school teachers; the main change would be a decrease in non-Euclidean models and greater attention to the high school topics mentioned above and including a transformational approach.

The number systems material in the mathematical structures course and the polynomials and other functions in the analysis course could be covered in a Foundations of High School Mathematics course that many departments offer, or it could be the focus of a capstone course. Alternatively, departments can use (and adjust) an existing number theory course that is geared to the needs of teachers. Readers are referred to the number theory section of the MET II report for a discussion of appropriate topics [3, p. 61].

The material in the analysis course can be incorporated into an existing analysis course, with particular attention to the topics highlighted in MET II.

For departments with an even further need to economize in courses, the transition to proof course can be focused so that its content is geared either to address material that fits in either the mathematical structures course or in the analysis course described above.

While the recommendations in this section represent our best vision of how to incorporate the required material into an existing program, we recognize the constraints on programs in smaller departments. We reiterate that our main recommendation in considering a program for teachers is to carefully address content that underlies, connects to, and is the focus of high school mathematics, either within existing courses required for all mathematics majors or in courses designed specifically for future teachers.

C. Lower-division related-area courses
This report recommends that mathematics teachers complete college-level courses in important areas where their students will use mathematics: computer science, physics, a second science course, and economics. Introduction to computer science and at least a semester of physics are common in existing mathematics teacher preparation programs.

A course in computer science is essential for many reasons. It teaches algorithmic thinking that is largely missing in mathematics courses but important in computer science, and which connects well with computational algorithms that are the focus of arithmetic in elementary grades. The course should include extensive programming, to help teachers understand how high school mathematics is an essential foundation for programming and computational thinking, in general. The recent trends in K-12 schooling include instruction in computer science, and mathematics teachers will often be asked to teach these courses.

The physics, a second science, and economics courses, which will normally count towards a college's general education requirement, enable teachers to enrich their instruction for future STEM and economics/business majors with motivating examples. The recommendation of an economics course is new to most mathematics teacher preparation programs, but is included because of the important quantitative aspects of the discipline.

D. Education-related courses and student teaching
A mathematics teaching major contains requirements beyond the mathematics content and related-area courses. Individual state licensing requirements preclude any one-size-fits-all program for this component . These suggestions contain those topics that prepare a high school teacher to understand issues of teaching and learning in high school classrooms, but such topics may be organized differently in different states. We recommend that the program include early field experiences in K-12 classrooms, a focus on written and oral communication, and a meaningful focus on diversity and equity.

Early field experience. The first field experience should come prior to formal admission to the teacher preparation program. The field experience should allow meaningful interactions with K – 12 students but does not need to involve practice teaching. The experience should be structured so that undergraduates have the opportunity to reflect on the practice of teaching and mathematical content.

Communication courses. We anticipate these requirements will be fulfilled within a university's general education requirement. Written and oral communication should be an integral part of all mathematics and education courses.

Diversity, equity, and multiculturalism. Experiences should specifically address teaching English language learners, contextually relevant or context embedded instruction, and issues of equity in American school systems. Mathematicians are encouraged to address issues of diversity, equity, and multiculturalism in the courses they teach, for example, by integrating a historical perspective of these issues into content courses, but they can also address them by fostering classroom climates that encourage all learners.

Methods of teaching high school mathematics. In states where the certificate spans both middle and high school grades, methods for teaching both grades 5-8 and 9-12 should be included, preferably in separate courses. Issues of classroom management, not limited to behaviorist approaches, must be addressed in a professional program. These may be included in a methods course, in a separate course, or in other courses in the program.

Student teaching. Two phases of student teaching are appropriate. The first should coincide with a methods of teaching mathematics course, to allow for close supervision during the first student teaching experience. The second phase should be a semester-long experience. It is beyond the scope of this report to recommend models for student teaching, but a model that involves close collaboration between a master teacher and the preservice teacher is encouraged.

Comment on technology. The use of technology in mathematics education is an important issue for a professional program to address. Technology in the classroom has both productive and cognitive uses. While it may be appropriate to include a course in educational technology, the cognitive uses of technology in mathematics teaching and learning must be addressed. The caution is that if technology is viewed as an add-on rather than an integral part of learning mathematics, pre-service teachers will not develop sufficient knowledge of how to teach mathematics with technology in a meaningful way. A course that deals only with uses of technology that increase a teacher's efficiency has limited value.

Alternative pathways for smaller departments. We reiterate that our main recommendation in considering a program for teachers is to carefully address content that underlies, connects to, and is the focus of high school mathematics, either within existing courses that all mathematics majors take or in courses designed entirely for future teachers. Small departments unable to implement the full recommendations of this report should design a program that includes single and multivariable calculus, a data-based statistics course, transition to proofs, and linear algebra. They should include upper division mathematics courses that include at least three core mathematics for teachers courses, chosen from

1. Abstract algebra (with a syllabus geared towards teachers)
2. Geometry (with a syllabus geared towards teachers)
3. Analysis (with a syllabus geared towards teachers)
4. Capstone course for teachers, focusing on mathematical structure in high school algebra, geometry, and analysis.

The program should include at least one further mathematics course, chosen from the five subjects listed in the upper-division list above.

Beyond the four year degree. A truly professional model would include a residency period. In such a model, a beginning mathematics teacher would work closely with a master teacher, examining elements of the new teacher's practice to be developed and improved. It is beyond the scope of these recommendations to detail such a residency program. The development of such a program relies on a shift in the U.S. educational system's view of the profession of teaching.

The program we describe here is for an undergraduate major that leads to a teaching credential. Master's programs designed for high school teachers and focus on mathematics content for teaching are an important resource, and cover many of the areas we address here, as well as others. Ongoing professional development in mathematics, whether via a master's program or not, is an important part of the profession of teaching. The MET II report has an extensive discussion of professional development for high school teachers.

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