Note on the Evaluation of $\int_0^x \frac{1}{1+t^{2^n}}\,dt$

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In [1], the integral

$$I = \int_0^x \frac{1}{1+t^m}\,dt$$

has been evaluated for $m = 1, 2, 3, 4, 5, 6, 8$, and 10. In an attempt to evaluate I for all values of m, we find that it is possible to obtain the general value of I when $m = 2^n (n \geqslant 2)$. The steps involved are as follows:

Step I: By factoring $(t^{2^n}+1)$, we get 2^{n-1} distinct quadratic factors; of these, 2^{n-2} quadratic factors are of the form $(t^2 + at + 1)$, and the other 2^{n-2} quadratic factors are of the form $(t^2 - at + 1)$, where a is one of 2^{n-2} numbers given by

$$\sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \cdots \pm \sqrt{2}}}}\ . \tag{1}$$

Note that the number of 2's in a is $(n-1)$.

Step II: On separating $\frac{1}{1+t^{2^n}}$ into partial fractions, we have

$$\frac{1}{1+t^{2^n}} = \sum_{k=1}^{2^{n-2}} \left[\frac{A_k t + B_k}{t^2 + a_k t + 1} + \frac{C_k t + D_k}{t^2 - a_k t + 1} \right]. \tag{2}$$

Multiply equation (2) by $(1 + t^{2^n})$ on both sides and take the limit as t tends to α, where

$$\alpha = \left[-a_k + i\sqrt{\left(4 - a_k^2\right)} \right]/2, \text{ to get}$$

$$\lim_{t \to \alpha} \left[\frac{(A_k t + B_k)(t^{2^n} + I)}{\left(t^2 + a_k t + 1\right)} \right] = 1.$$

Note that $1 + \alpha^{2^n} = 0$, and $\alpha^2 + a_k \alpha + 1 = 0$.

Evaluating the limit and rearranging the result, we see that

$$2 + a_k \alpha - 2^n (A_k \alpha + B_k) = 0. \tag{3}$$

Equating the real and imaginary parts of (3) to zero and solving the resulting equations, it is seen that

$$A_k = a_k 2^{-n}, \text{ and } B_k = 2^{1-n}. \tag{4}$$

Similarly

$$C_k = -a_k 2^n, \text{ and } D_k = 2^{1-n}. \tag{5}$$

Substituting (4) and (5) in (2) and integrating both sides with respect to t between the limits zero and x, we get

$$\int_0^x \frac{1}{1+t^{2^n}}\,dt = \sum_{k=1}^{2^{n-2}} \frac{a_k}{2^{n+1}}\left[\log\left(\frac{x^2+a_k x+1}{x^2-a_k x+1}\right)+2\tan^{-1}\left(\frac{a_k x}{1-x^2}\right)\right]. \qquad (6)$$

Examples

(1) For $n=2$, the number of 2's in a is 1 and $a_1=\sqrt{2}$.
From (6), we get

$$\int_0^x \frac{1}{1+t^4}\,dt = \frac{1}{2\sqrt{2}}\left[\log\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+2\tan^{-1}\left(\frac{\sqrt{2}x}{1-x^2}\right)\right].$$

(2) For $n=3$, the number of 2's in a is 2 and therefore

$$a_1=\sqrt{2+\sqrt{2}}\,,\ \text{and}\ a_2=\sqrt{2-\sqrt{2}}\,.$$

(3) For $n=4$, the number of 2's in a is 3 and

$$a_1=\sqrt{2+\sqrt{2+\sqrt{2}}}\,,\ a_2=\sqrt{2+\sqrt{2-\sqrt{2}}}\,,$$

$$a_3=\sqrt{2-\sqrt{2+\sqrt{2}}}\,,\ \text{and}\ a_4=\sqrt{2-\sqrt{2-\sqrt{2}}}\,.$$

It is seen that Examples (1) and (2) are in agreement with [1].

REFERENCE

1. *Notebooks of Srinivasa Ramanujan*, Vol. 2, Tata Institute of Fundamental Research, Bombay, 1957.