An Alternate Proof of Cramer's Rule

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In almost every introductory book on linear algebra, the proof of Cramer's Rule assumes that students are familiar with the classical adjoint, adj A, of a matrix A. The proof then uses the result that $A(\operatorname{adj} A) = (\det A)I$. In their text *Matrix Analysis* [Cambridge University Press, New York, 1985, p. 21], Roger A. Horn and Charles A. Johnson sketch an elementary proof that we believe should be the standard of Cramer's Rule in linear algebra texts. Our intent is to call attention to their proof and fill in sufficiently many details so that the proof may be presented in a first course in linear algebra.

Let $C = (C_1, C_2, \dots, C_n)$ be an $n \times n$ matrix whose ith column is C_i, and let $C(y; i)$ denote the $n \times n$ matrix obtained from C by replacing C_i by the column vector y. Using this notation, Cramer's Rule asserts the following:

> *Let A be an $n \times n$ nonsingular matrix. Then the (unique) solution to $Ax = b$ is given by the vector $x = (x_1, \dots, x_n)$, whose ith coordinate is*
>
> $$x_i = \frac{\det A(b; i)}{\det A}.$$

Let $I(x; i) = (e_1, e_2, \dots, e_{i-1}, x, e_{i+1}, \dots, e_n)$ denote the $n \times n$ identity matrix with the ith column e_i replaced by the column vector x. Then

$$\begin{aligned} AI(x; i) &= A(e_1, e_2, \dots, e_{i-1}, x, e_{i+1}, \dots, e_n) \\ &= (Ae_1, Ae_2, \dots, Ae_{i-1}, Ax, Ae_{i+1}, \dots, Ae_n) \\ &= (A_1, A_2, \dots, A_{i-1}, b, A_{i+1}, \dots, A_n) \\ &= A(b; i). \end{aligned}$$

Therefore,

$$\det A \det I(x; i) = \det A(b; i).$$

But $\det I(x; i) = x_i$ since the determinant of a lower triangular matrix is the product of its diagonal entries.

How Many Bridge Auctions?

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In normal games of bridge, only a small fraction of the possible auctions (that is, sequences of bids and passes) are ever used. One might therefore be surprised at the actual number of distinct auctions.

The game of bridge is played with four players comprising two teams, North-South and East-West. We will assume that North opens the bidding, which proceeds clockwise. Each player bids or passes. The 35 possible bids are (in increasing order) 1 club, 1 diamond, 1 heart, 1 spade, 1 no trump; 2 clubs, ..., 2 no trump; ...; 7 clubs, ..., 7 no trump. Each bid must be higher than the preceding bid. The bidding terminates if all four players pass on the first round, or when 3 consecutive passes occur after the first bid. In place of a pass or bid, a player may "double" the last bid provided that this bid was not made by his partner. For example, South cannot double North's bid if East passes. Following a double, a player may "redouble" provided that he does not redouble his partner's double. There can be no more doubling of a bid after a redouble is made, unless a subsequent bid is made—in which case, it too may be doubled and then redoubled.