Theorem 3 can be proved much like Theorem 2. For Theorem 4, we ask the reader to verify the first few cases for $k = 2, 3, \ldots$. It would be interesting to get a closed form for g. Finding other pairs of functions that constitute PP-2 pairs (for example in the case when f is a trigonometric function) is also an interesting challenge!

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The Right Theta

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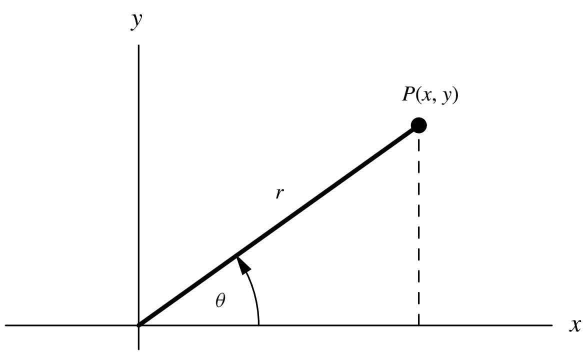
A basic concept in trigonometry is the angle θ associated with a point $P(x, y)$ in the plane. The exact elementary formula for the angle function $\theta = \theta(x, y)$ has a complicated case-by-case description. In this capsule we derive a more concise formula which applies to the entire Cartesian plane (except the origin and points on the negative x-axis).



The Cartesian plane.

As usual, we measure θ positive counterclockwise from the positive x-axis and take $-\pi < \theta \leq \pi$ radians. The graph of this function is shown in Figure 1 (see Note 2 on how this graph was constructed).

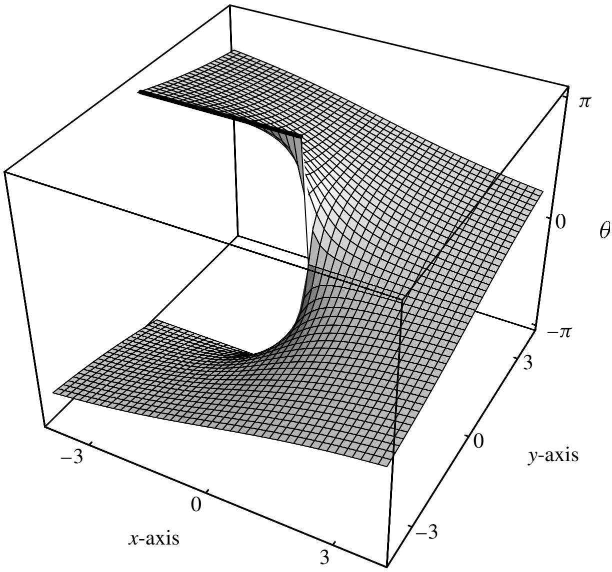


Figure 1. The angle θ as a function of x and y.

In elementary courses the value of the angle function is usually presented using the formula $\theta = \tan^{-1}\frac{y}{x}$. The graph of this formula, shown in Figure 2, is correct only in the right half-plane. One must add or subtract π in the second or third quadrants. In addition, it is necessary to make appropriate choices of its values on the negative x-axis and the positive and negative y-axes. θ is left undefined at the origin.

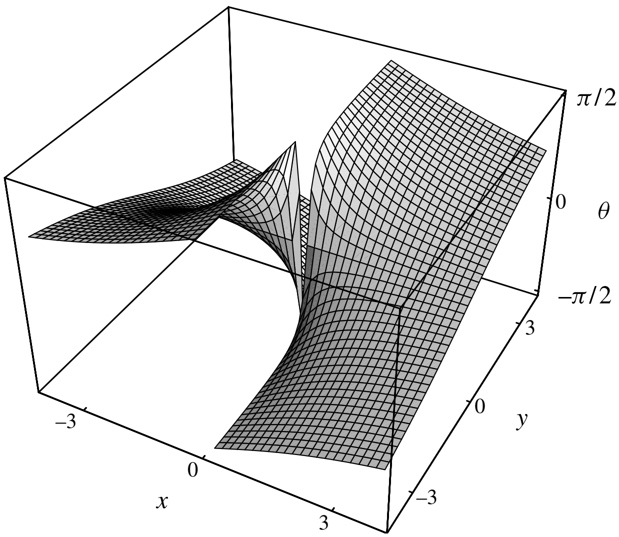


Figure 2. An incorrect theta function.

The complete form of θ for all $(x, y) \neq (0, 0)$ in terms of $\tan^{-1}\frac{y}{x}$ is then

$$\theta = \begin{cases} \tan^{-1}\frac{y}{x}, & \text{for } x > 0 \\ \tan^{-1}\frac{y}{x} + \pi, & \text{for } x < 0 \text{ and } y > 0 \\ \tan^{-1}\frac{y}{x} - \pi, & \text{for } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2}, & \text{for } x = 0 \text{ and } y > 0 \\ \pi, & \text{for } x < 0 \text{ and } y = 0 \\ -\frac{\pi}{2}, & \text{for } x = 0 \text{ and } y < 0. \end{cases} \tag{1}$$

The graph of this piecewise-defined function is shown in Figure 3. As a visual check, compare with Figure 1 that this is the same θ function.

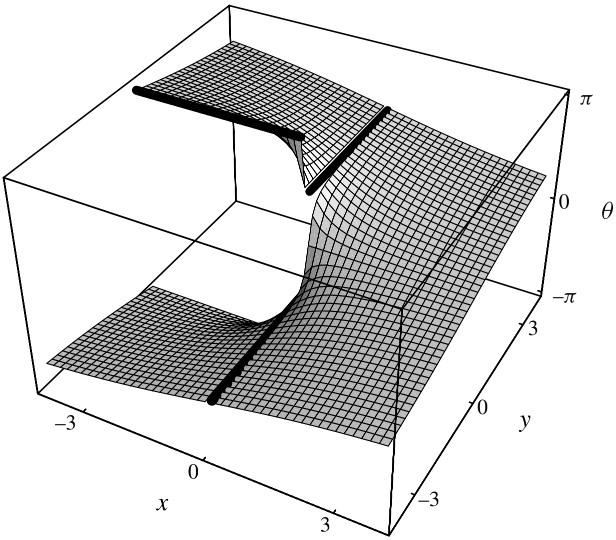


Figure 3. The piecewise-defined theta function from (1).

Let us derive a simpler formula valid in a larger domain. We begin with the half-angle formulas

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} \quad \text{and} \quad \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}.$$

Using a little trigonometry, we deduce that

$$\tan^2\frac{\theta}{2} = \frac{\sin^2\theta}{(1+\cos\theta)^2}.$$

Taking a square root (see Note 4), using the relationships $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$ and $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$, and solving for θ we get

$$\boxed{\theta = 2\tan^{-1}\frac{y}{x+\sqrt{x^2+y^2}}} \tag{2}$$

which is defined for all points in the plane except at the origin and along the negative x-axis.

The complete form of θ in terms of the preceding formula is therefore

$$\theta = \begin{cases} 2\tan^{-1}\frac{y}{x+\sqrt{x^2+y^2}}, & \text{except for } x < 0 \text{ and } y = 0 \\ \pi, & \text{for } x < 0 \text{ and } y = 0. \end{cases} \tag{3}$$

The graph of this function agrees with Figure 1, but this time uses formula (2). Note that $2\tan^{-1}\frac{y}{x+\sqrt{x^2+y^2}}$ is undefined at the origin and also along the negative x-axis where its argument has the indeterminate form $\frac{0}{0}$. This agrees with the fact that θ could reasonably be defined as either π or $-\pi$ there.

We conclude this paper with a few notes.

1. It is unlikely formula (2) is new because the idea is so elementary. However, in our many years of teaching, neither of us has met it. Regardless, it is not well known and may well be useful because, even though the argument in the arctan formula involved is more complicated, the logical structure of the complete formula is much simpler.
2. The graph of Figure 1 was produced by the *Mathematica* function ArcTan[x, y], not to be confused with $\tan^{-1}\frac{y}{x}$. ArcTan[x, y] is numerically evaluated for non-zero real numbers by calling the C library function atan2(x, y), which is likely an implementation of (1). (Readers may wish to check how θ is calculated in the symbolic algebra programs they use.)
3. Symbolic algebra programs may not yet use (2). For example, for the point (1, 1), *Mathematica* gives the correct value of ArcTan $\frac{1}{1}$ as $\frac{\pi}{4}$ (likely using a lookup table), but leaves 2 ArcTan $\frac{1}{1+\sqrt{1^1+1^2}}$ unevaluated.
4. In our derivation of (2) it is possible to choose the other square root. If this is done, then the angle range is $-\pi < \theta < \pi$ described clockwise. A more useful variation, left as an exercise, is $\theta = 180 + \frac{360}{\pi}\tan^{-1}\frac{y}{x-\sqrt{x^2+y^2}}$. For this formula, $0 < \theta < 360°$, and is measured positive counterclockwise from the positive x-axis. Its graph is shown in Figure 4.
5. $2\tan^{-1}\frac{y}{x+\sqrt{x^2+y^2}}$ is a potential function for the line integral

   $$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy,$$

   where C is any smooth simple closed path about the origin. Then, by Figure 1, in a manner reminiscent of the evaluation of closed contour integrals in complex variables, it is clear that

   $$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \pi - (-\pi^+) = 2\pi.$$

   $\tan^{-1}\frac{y}{x}$ is also an antiderivative; however, Figure 2 is difficult to interpret for this purpose. The downside of using our formula (2) is that it is very difficult to show by hand, either by differentiation or by integration, that it is a potential function.

This line integral is an example in all five calculus textbooks that we looked at (Stewart; Edwards and Penny; Thomas; Rogawski; and Anton, Bivens, and Davis).

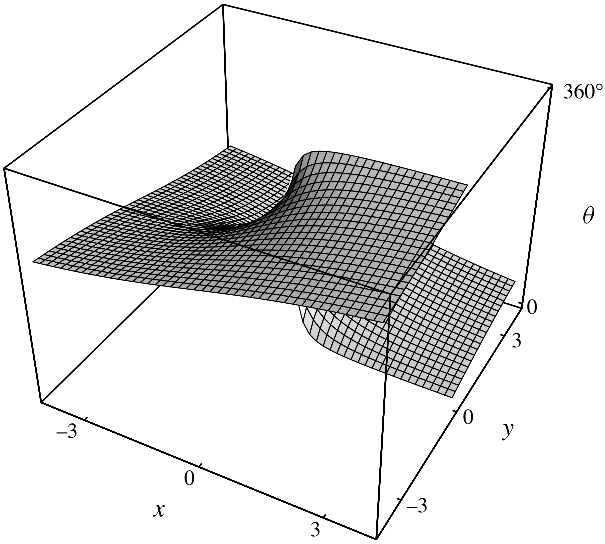


Figure 4. Another theta function useful for some applications.

Some textbooks evaluate it directly as a line integral about a circle in the section on Green's Theorem. Others, in addition, show that $\tan^{-1}\frac{y}{x}$ is a potential function and note it can be interpreted geometrically as the angle function θ, and that therefore the change of angle along a closed contour about the origin is 2π. None comment on the difficulties in interpreting this expression in the entire plane or graph this potential function in the entire plane, perhaps to avoid having to explain the complexities of formula (1).

Ken Yanosko (ky1@humboldt.edu), Humboldt State University, wrote that he enjoyed reading the article "Commensurable Triangles" by Richard Parris in the November issue (38 (2007) 345–355), but points out earlier work on the topic. R. S. Luthar classified 2-commensurable triangles (Integer-Sided Triangles with One Angle Twice Another, *College Math. J.* 15 (1984) 55–560), and subsequently, the general case in which one angle is a rational multiple of another was treated by J. Carroll and K. Yanosko, (The Determination of a Class of Primitive Integral Triangles, *Fibonacci Quarterly* 29 (1991) 3–6).

Robert Clay (rclay@daltonstate.edu), Dalton State College, Dalton, Georgia, writes concerning the Classroom Capsule "Conic Sections from the Plane Point of View" by Sidney Kung in the November 2007 issue (pp. 383–384): The article is correct in stating that equation (3) is a conic section. However, it is not congruent to the original conic formed by the cutting plane, but rather the projection of the original onto the xy-plane. The original and the projection will be congruent if the cutting plane is parallel to the xy-plane. Therefore, one can achieve congruence by rotating the cone and the cutting plane until the plane is parallel to the xy-plane.