For $(x, y) = (4, 2)$, we have $g'(u)/g(u) = \frac{4-6u}{3u^2-4u+1}$, so the expected number of flips is

$$4 + 2pq \cdot \frac{4 - 6pq}{3p^2q^2 - 4pq + 1}.$$

For $p = q = \frac{1}{2}$ and $u = \frac{1}{4}$, the expected time to lose is $32/3$.

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Summary In a version of gambler's ruin, players start with x and y dollars respectively, and flip coins for one dollar per flip until one player runs out of money. This is a random walk with two absorbing barriers. We consider the number of ways for the first player to lose on the nth flip, for $n = x, x + 2, \ldots$. We use probabilistic arguments to construct generating functions for these quantities along with explicit methods for computing them. This paper builds on the paper by Hirshon and De Simone, *Mathematics Magazine* **81** (2008) 146–152.

More Polynomial Root Squeezing

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Suppose you're looking at the graph of a polynomial $y = p(x)$ in a java applet, with blue dots on the x-axis indicating the polynomial's roots, and red dots on the x-axis showing the positions of the critical points. Let's assume that all the roots are real and that you grab the blue dots and move them around on the x-axis. As you do this, what happens to the red dots?

This is a fair question because the roots determine the polynomial up to a constant multiple, and they determine the critical points exactly. For simplicity (and without loss of generality) we will only consider monic polynomials (that is, polynomials with leading coefficient 1).

If you move all the blue points (roots) the same amount, the whole graph just translates, and all the red dots simply move along for the ride. If you move all the roots in the same direction but by different amounts, it seems reasonable that the critical points all move in that same direction. This is in fact true, according to the Polynomial Root Dragging Theorem (see [**1**], [**3**]). But suppose you take two roots and symmetrically squeeze them closer to each other, something we call polynomial root squeezing. Then

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what do the critical points do? In [**2**], Boelkins, From and Kolins answer this for critical points that are outside the interval between the two selected roots. In this article we extend their analysis to cover critical points at or between the two squeezed roots.

Notation and definitions Let $p(x)$ be a monic degree-n polynomial with real roots $r_1 \le r_2 \le \cdots \le r_n$ and critical points $c_1 \le c_2 \le \cdots \le c_{n-1}$. Rolle's Theorem tells us that there is a critical point strictly between each pair of adjacent roots. We know that wherever there are r roots together at a single point, there are also $(r-1)$ critical points. So we have

$$r_1 \le c_1 \le r_2 \le c_2 \le \cdots \le c_{n-1} \le r_n \tag{1}$$

with $r_i < c_i < r_{i+1}$ whenever $r_i < r_{i+1}$. By polynomial root squeezing we mean selecting two indices i and j with r_i strictly less than r_j; we then move the smaller root from r_i to $r_i + d$ and the larger root from r_j to $r_j - d$, where $d > 0$. We insist that $d < \frac{r_i + r_j}{2}$, so that the roots don't pass each other.

As an example, consider the polynomial $p(x) = x^2(x+1)(x-2)$. It has single roots at -1 and 2, and a double root at 0. Its critical points are at (approximately) $-.693$, 0, and 1.443. After squeezing the roots at -1 and 2 to $-.5$ and 1.5 respectively, the polynomial becomes $\tilde{p}(x) = x^2(x+.5)(x-1.5)$. The left critical point moves to the right from $-.693$ to $-.343$, and the right critical point moves to the left from 1.443 to 1.093. However the center critical point remains at zero. This example is illustrated in FIGURE 1.

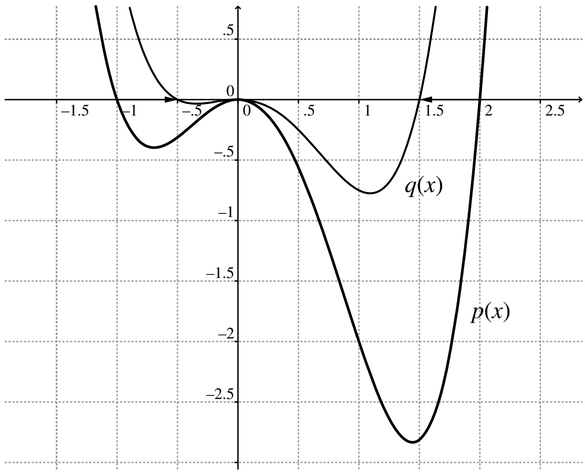

Figure 1 Two roots of the polynomial $p(x) = x^2(x+1)(x-2)$ have been squeezed together to form $\tilde{p}(x)$. In this example, $x = 0$ is a critical point of $p(x)$ and $q(x)$.

Why doesn't the critical point at zero move? It is because $x = 0$ is a repeated root of $\frac{p(x)}{(x+1)(x-2)}$, and as long as this repeated root remains fixed, so must the critical point. More generally, if c_k is a repeated root of $\frac{p(x)}{(x-r_i)(x-r_j)}$, then c_k will remain a critical point when r_i and r_j are squeezed together. For this reason, we say that a critical point is **stubborn** if it is a repeated root of $\frac{p(x)}{(x-r_i)(x-r_j)}$, and **ordinary** otherwise.

A stubborn critical point can move if it lies at r_i or r_j. If r_i (or r_j) lies at a repeated root of multiplicity greater than two, then there is a repeated stubborn critical point there. When r_i is dragged to the right, one of the stubborn critical points will move to the right, while the others will remain fixed. In order to state the theorem as succinctly as possible we exclude the case of stubborn critical points and leave the details as an exercise.

The theorem Boelkins, From and Kolins [**2**] proved the Polynomial Root Squeezing Theorem. That theorem explains how squeezing two roots together affects the critical points that are outside of the interval between the two squeezed roots. Our proof of the Polynomial Root Squeezing Theorem extends their analysis to the critical points that lie at or between the two squeezed roots.

THEOREM. *If the roots at r_i and r_j move equal distances toward each other, then each ordinary critical point moves toward $(r_i + r_j)/2$. If the roots at r_i and r_j move equal distances away from each other, then each ordinary critical point moves away from $(r_i + r_j)/2$.*

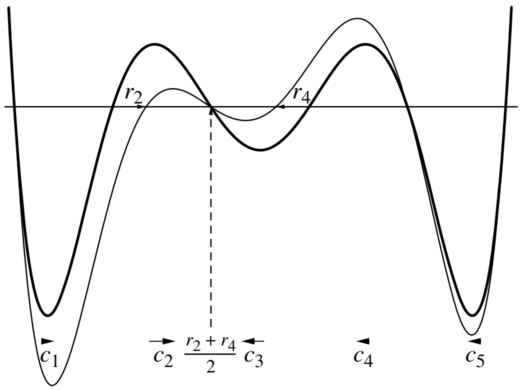

Figure 2 The Polynomial Root Squeezing Theorem: when we drag r_2 and r_4 together, the critical points move toward $(r_2 + r_4)/2$.

Proof. We prove the root squeezing part of the theorem. The root separating part (moving r_i and r_j equal distances away from each other) follows similarly.

Let $p(x)$ be a polynomial of degree n with (possibly repeated) real roots $r_1 \leq r_2 \leq \cdots \leq r_n$, $r_i < r_j$ and c_k any critical point of $p(x)$. Let $\tilde{p}(x)$ be the polynomial that results from squeezing r_i and r_j a fixed distance d, with $0 \leq d < \frac{1}{2}\left(r_j - r_i\right)$. That is

$$\begin{aligned}\tilde{p}(x) &= (x - r_i - d)(x - r_j + d) \prod_{k \neq i,j} (x - r_k) \\ &= (x - r_i - d)(x - r_j + d)q(x).\end{aligned}$$

Denote the roots of $\tilde{p}(x)$ by $\tilde{r}_1 \leq \tilde{r}_2 \leq \ldots \leq \tilde{r}_n$ and the critical points by $\tilde{c}_1 \leq \tilde{c}_2 \leq \ldots \leq \tilde{c}_{n-1}$.

If c_k lies outside the interval from r_i to r_j, then the conclusion follows from [**2**]. (It also follows from a slight variation of the reasoning below.) If c_k is between r_i and $r_i + d$, or between $r_j - d$ and r_j (that is, if one of the moving roots passes by c_k) then the result follows from counting intervals in (1).

We now assume that c_k is not at a repeated root of p and that $r_i + d < c_k < r_j - d$. Our goal is to compare c_k and $\tilde{c_k}$. We do so by investigating $\tilde{p}'(c_k)$. Let

$$p(x) = (x - r_i)(x - r_j)q(x),$$

so that

$$p'(x) = (x - r_i + x - r_j)q(x) + (x - r_i)(x - r_j)q'(x), \tag{2}$$

and

$$\tilde{p}'(x) = (x - r_i + x - r_j)q(x) + (x - r_i - d)(x - r_j + d)q'(x). \quad (3)$$

Subtracting (2) from (3) yields

$$\tilde{p}'(c_k) = d(r_j - r_i - d)q'(c_k). \quad (4)$$

Since $r_j - r_i - d > 0$, this implies that $\tilde{p}'(c_k)$ and $q'(c_k)$ have the same sign.

Without loss of generality we assume that $p(x) < 0$ on (r_k, r_{k+1}) and that $|c_k - r_i| < |c_k - r_j|$ (The cases where $|c_k - r_i| > |c_k - r_j|$ and or $p(x) > 0$ are similar.) Since $r_i < c_k < r_j$, it follows that $(c_k - r_i)(c_k - r_j) < 0$ so that $q(c_k) > 0$. As $p'(c_k) = 0$,

$$0 = p'(c_k) = (c_k - r_i + c_k - r_j)q(c_k) + (c_k - r_i)(c_k - r_j)q'(c_k).$$

An analysis of the sign of the terms, with the assumption that $|c_k - r_i| < |c_k - r_j|$, implies that $q'(c_k) < 0$. It then follows from (4) that $\tilde{p}'(c_k) < 0$.

Since $p(c_k) < 0$, the equation

$$p(c_k)(c_k - r_i - d)(c_k - r_j + d) = \tilde{p}(c_k)(c_k - r_i)(c_k - r_j)$$

implies that $\tilde{p}(c_k) < 0$. Since we assume that $r_i + d < c_k < r_j - d$ and c_k is not a repeated root of p, it follows that $\tilde{r}_k = r_k$ or $\tilde{r}_k = r_i + d$ while $\tilde{r}_{k+1} = r_{k+1}$ or $\tilde{r}_{k+1} = r_j - d$. In all four cases, $\tilde{r}_k < c_k < \tilde{r}_{k+1}$ with $\tilde{p}(c_k) < 0$ which implies that $\tilde{p}(x) < 0$ on $(\tilde{r}_k, \tilde{r}_{k+1})$. Therefore $\tilde{p}'(x)$ changes sign from negative to positive at $\tilde{c}_k$. As $\tilde{p}'(c_k) < 0$, it follows that $c_k < \tilde{c}_k$ and $\tilde{c}_k$ has moved toward $(r_i + r_j)/2$. ■

This extended version of the Polynomial Root Squeezing Theorem completely characterizes the behavior of all the critical points when distinct roots are squeezed or separated a uniform distance. In every case, if a critical point moves at all, it moves in the same direction as the moving root that is nearest to it.

Unfortunately, this intuition does not help us when two distinct roots are squeezed together a nonuniform distance. Neither does it tell us what happens when more than two roots are moved simultaneously. These problems could prompt some interesting undergraduate research.

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Summary Given a polynomial with all real roots, the Polynomial Root Dragging Theorem states that moving one or more roots of the polynomial to the right will cause every critical point to move to the right, or stay fixed. But what happens to the position of a critical point when roots are dragged in opposite directions? In this note we discuss the Polynomial Root Squeezing Theorem, which states that moving two roots, r_i and r_j, an equal distance toward each other without passing other roots, will cause each critical point to move toward $(r_i + r_j)/2$, or remain fixed.