Finding Curves with Computable Arc Length

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One of the standard applications of the integral in elementary calculus is to find the arc length of a curve. Having derived the formula

$$L = \int_a^b \sqrt{(f'(x))^2 + 1}\,dx \tag{1}$$

for the length L of the graph of the differentiable function $f(x)$ for $a \le x \le b$, as instructors we often search for examples for which the integral has a nice closed form. Because of that awkward square root, it is apparent that for most elementary functions, the integral in (1) will not be pleasant.

Most calculus books give a few examples of functions for which the integral can be evaluated. The function $f(x) = x^{3/2}$ is a fairly standard one, but more interesting is $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ ([**1**, p. 398]). In this case, $(f'(x))^2 + 1$ turns out to equal $(\frac{x^2}{2} + \frac{1}{2x^2})^2$, from which it follows that the indefinite integral $\int \sqrt{(f'(x))^2 + 1}\,dx$ is just $\frac{x^3}{6} - \frac{1}{2x} + C$. The point, of course, is that $(f'(x))^2 + 1$ is a perfect square. Other such functions in textbooks are $f(x) = \frac{x^4}{8} + \frac{1}{4x^2}$ [**1**, p. 402] and $f(x) = \frac{x^2}{8} - \ln x$ [**2**, p. 465].

While searching for more examples when I was teaching calculus recently, I happened upon an algorithm that produces functions with that "perfect square property." This algorithm is almost surely known to someone, but I have not been able to find it anywhere. Assume that $(f'(x))^2 + 1 = (s(x))^2$. Then it must be the case that $(s(x) + f'(x))(s(x) - f'(x)) = 1$. Now let $g(x) = s(x) + f'(x)$, whence $\frac{1}{g(x)} = s(x) - f'(x)$, and so $f'(x) = \frac{1}{2}\left(g(x) - \frac{1}{g(x)}\right)$. Consequently,

$$f(x) = \frac{1}{2}\int \left(g(x) - \frac{1}{g(x)}\right)\,dx. \tag{2}$$

Thus, we can take $g(x)$ to be almost any function we like (as long as we have differentiability of $f(x)$ on our interval $[a, b]$). Observe that, with $f(x)$ as in (2), (1) becomes

$$L = \frac{1}{2}\int_b^a \left(g(x) + \frac{1}{g(x)}\right)\,dx. \tag{3}$$

For example, if $g(x) = x^{10}$, then

$$f(x) = \frac{1}{2}\int \left(x^{10} - \frac{1}{x^{10}}\right) dx = \frac{x^{11}}{22} + \frac{1}{18x^9} \ (+C).$$

Or try $g(x) = \tan x$. Then the indefinite integral for $f(x)$ can be computed, using some trig identities, as

$$\begin{aligned}\frac{1}{2}\int (\tan x - \cot x)\, dx &= \frac{1}{2}\left(-\ln\left(\frac{1}{2}\sin 2x\right)\right) + C \\ &= -\frac{1}{2}\ln(\sin 2x) - \frac{1}{2}\ln\left(\frac{1}{2}\right) + C.\end{aligned}$$

By ignoring the constants, we can choose $f(x) = \frac{1}{2}\ln(\sin 2x)$. However, although $f'(x)^2 + 1$ will not then equal $\left(\frac{1}{2}\left(g(x) + \frac{1}{g(x)}\right)\right)^2$, things still come out nicely:

$$\begin{aligned}\int \sqrt{(f'(x))^2 + 1}\, dx = \int \sqrt{\cot^2 2x + 1}\, dx &= \int \csc 2x\, dx \\ &= -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C \quad (\text{for } 0 \le x \le \tfrac{\pi}{2}).\end{aligned}$$

We invite the reader to experiment with this algorithm and discover other examples.

References

1. C. H. Edwards and D. E. Penney, *Calculus*, 6th ed., Prentice Hall, 2002.
2. G. B. Thomas, Jr., R. L. Finney, M. D. Weir, and F. R. Giordano, *Thomas' Calculus*, 10th ed., Addison-Wesley, 2001.

Arc Length and Pythagorean Triples

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In this note we give an example of how a computer algebra system can offer surprises even in the context of a standard calculus topic. When introducing the formula for arc length, some natural examples are the curves C_n which are given parametrically by $x = t^n$, $y = t^{n+1}$, $0 \le t \le 1$, (n is a positive integer). Many students have difficulty computing even the length of C_1 by hand, so this is a natural place to use a computer algebra system. The length of C_5, for example, is

$$\frac{3431\sqrt{61}}{20736} + \frac{15625}{124416}\ln 5 - \frac{15625}{124416}\ln(-6 + \sqrt{61}).$$

As n increases, the results become increasingly unpleasant until, surprisingly, we find that the length of C_{20} is rational and equals $\frac{36495661067145135829027}{25798674916142804999323}$.