

CLASSROOM CAPSULES

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Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editors.

Finding Curves with Computable Arc Length

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One of the standard applications of the integral in elementary calculus is to find the arc length of a curve. Having derived the formula

$$L = \int_a^b \sqrt{(f'(x))^2 + 1} dx \quad (1)$$

for the length L of the graph of the differentiable function $f(x)$ for $a \leq x \leq b$, as instructors we often search for examples for which the integral has a nice closed form. Because of that awkward square root, it is apparent that for most elementary functions, the integral in (1) will not be pleasant.

Most calculus books give a few examples of functions for which the integral can be evaluated. The function $f(x) = x^{3/2}$ is a fairly standard one, but more interesting is $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ ([1, p. 398]). In this case, $(f'(x))^2 + 1$ turns out to equal $(\frac{x^2}{2} + \frac{1}{2x^2})^2$, from which it follows that the indefinite integral $\int \sqrt{(f'(x))^2 + 1} dx$ is just $\frac{x^3}{6} - \frac{1}{2x} + C$. The point, of course, is that $(f'(x))^2 + 1$ is a perfect square. Other such functions in textbooks are $f(x) = \frac{x^4}{8} + \frac{1}{4x^2}$ [1, p. 402] and $f(x) = \frac{x^2}{8} - \ln x$ [2, p. 465].

While searching for more examples when I was teaching calculus recently, I happened upon an algorithm that produces functions with that “perfect square property.” This algorithm is almost surely known to someone, but I have not been able to find it anywhere. Assume that $(f'(x))^2 + 1 = (s(x))^2$. Then it must be the case that $(s(x) + f'(x))(s(x) - f'(x)) = 1$. Now let $g(x) = s(x) + f'(x)$, whence $\frac{1}{g(x)} = s(x) - f'(x)$, and so $f'(x) = \frac{1}{2} \left(g(x) - \frac{1}{g(x)} \right)$. Consequently,

$$f(x) = \frac{1}{2} \int \left(g(x) - \frac{1}{g(x)} \right) dx. \quad (2)$$

Thus, we can take $g(x)$ to be almost any function we like (as long as we have differentiability of $f(x)$ on our interval $[a, b]$). Observe that, with $f(x)$ as in (2), (1) becomes

$$L = \frac{1}{2} \int_b^a \left(g(x) + \frac{1}{g(x)} \right) dx. \quad (3)$$

For example, if $g(x) = x^{10}$, then

$$f(x) = \frac{1}{2} \int \left(x^{10} - \frac{1}{x^{10}} \right) dx = \frac{x^{11}}{22} + \frac{1}{18x^9} (+C).$$

Or try $g(x) = \tan x$. Then the indefinite integral for $f(x)$ can be computed, using some trig identities, as

$$\begin{aligned} \frac{1}{2} \int (\tan x - \cot x) dx &= \frac{1}{2} \left(-\ln \left(\frac{1}{2} \sin 2x \right) \right) + C \\ &= -\frac{1}{2} \ln(\sin 2x) - \frac{1}{2} \ln \left(\frac{1}{2} \right) + C. \end{aligned}$$

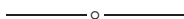
By ignoring the constants, we can choose $f(x) = \frac{1}{2} \ln(\sin 2x)$. However, although $f'(x)^2 + 1$ will not then equal $\left(\frac{1}{2} \left(g(x) + \frac{1}{g(x)} \right) \right)^2$, things still come out nicely:

$$\begin{aligned} \int \sqrt{(f'(x))^2 + 1} dx &= \int \sqrt{\cot^2 2x + 1} dx = \int \csc 2x dx \\ &= -\frac{1}{2} \ln |\csc 2x + \cot 2x| + C \quad (\text{for } 0 \leq x \leq \frac{\pi}{2}). \end{aligned}$$

We invite the reader to experiment with this algorithm and discover other examples.

References

1. C. H. Edwards and D. E. Penney, *Calculus*, 6th ed., Prentice Hall, 2002.
2. G. B. Thomas, Jr., R. L. Finney, M. D. Weir, and F. R. Giordano, *Thomas' Calculus*, 10th ed., Addison-Wesley, 2001.



Arc Length and Pythagorean Triples

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In this note we give an example of how a computer algebra system can offer surprises even in the context of a standard calculus topic. When introducing the formula for arc length, some natural examples are the curves C_n which are given parametrically by $x = t^n$, $y = t^{n+1}$, $0 \leq t \leq 1$, (n is a positive integer). Many students have difficulty computing even the length of C_1 by hand, so this is a natural place to use a computer algebra system. The length of C_5 , for example, is

$$\frac{3431\sqrt{61}}{20736} + \frac{15625}{124416} \ln 5 - \frac{15625}{124416} \ln(-6 + \sqrt{61}).$$

As n increases, the results become increasingly unpleasant until, surprisingly, we find that the length of C_{20} is rational and equals $\frac{36495661067145135829027}{25798674916142804999323}$.