Lognormal density This density usefully represents the distribution of size for varied kinds of 'natural' economic or physical units. The case of the lognormal requires somewhat more mathematical manipulation than the previous two examples. Imagine these computations when using the original function $f(x)$! The pdf of the lognormal is

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi}} e^{-(\ln x-\mu)^2/2\sigma^2} \qquad x > 0,$$

where $\mu \in R^1$and $\sigma > 0$ are the location and scale parameters. Now, by taking the natural logarithm of $f(x)$ in the customary fashion, we get

$$g(x) = \ln f(x) = -\ln\sigma - \ln x - \ln\sqrt{2\pi} - (\ln x - \mu)^2/2\sigma^2.$$

Differentiation yields

$$g'(x) = -\frac{1}{x} - \frac{(\ln x - \mu)}{\sigma^2 x},$$

from which we find that the critical value is $x = e^{\mu-\sigma^2}$. Differentiating again, we get

$$g''(x) = \frac{1}{x^2} - \frac{1}{\sigma^2}\left\{\frac{1 - \ln x + \mu}{x^2}\right\},$$

which is negative at the critical value. This confirms the existence of the mode at $x = e^{\mu-\sigma^2}$. In conclusion, we note that the method given here can also be used to locate inflection points of the density curve by solving the equation

$$f''(x) = e^{g(x)}g''(x) + e^{g(x)}[g'(x)]^2 = 0.$$

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Searching for Möbius

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The Möbius function μ, defined on the integers by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is divisible by a square} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes,} \end{cases}$$

is ubiquitous in number theory. Like many other natural mathematical objects, its definition seems at first glance to be contrived. Apostol [**1**] shows how it arises naturally as the identity for Dirichlet convolution, a binary operation on arithmetic functions. In this note, we develop it another way, through the algebra of formal series.

A formal Dirichlet series is an expression of the form

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where the $a(n)$ are complex numbers. The word "formal" is important here—we think of these series as bookkeeping mechanisms to keep track of combinatorial or numerical data. So, we do not worry about questions of convergence, and we think of s simply as an *indeterminate* (rather than a variable that can be replaced by a real or complex number). This misses many of the wonderful analytic applications of such series, and, as we are warned by Wilf [**2**],

> . . . [t]o omit those parts of the subject, however, is like listening to a stereo broadcast of, say, Beethoven's Ninth Symphony, using only the left audio channel.

But it turns out that the left channel is all we need for this discussion.

In addition to offering an alternative motivation for the definition of μ, another goal of this paper is to lobby for the integration of some formal algebra and combinatorics into courses in elementary number theory. This idea isn't new (see [**3**] or [**4**], for example), but the increasing use of CAS technology outside of calculus might provide some impetus for it.

Dirichlet series are added and multiplied formally. Addition is done term by term:

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} + \sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \sum_{n=1}^{\infty} \frac{a(n)+b(n)}{n^s}. \tag{1}$$

Multiplication is also done term by term, but then one gathers up all terms with the same denominator. So, for example, suppose we are looking for $c(12)/12^s$ in

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} \sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}.$$

Then a denominator of 12^s could only come from the products

$$\frac{a(1)}{1^s}\cdot\frac{b(12)}{12^s},\ \frac{a(2)}{2^s}\cdot\frac{b(6)}{6^s},\ \frac{a(3)}{3^s}\cdot\frac{b(4)}{4^s},\ \frac{a(4)}{4^s}\cdot\frac{b(3)}{3^s},\ \frac{a(6)}{6^s}\cdot\frac{b(2)}{2^s},\ \frac{a(12)}{12^s}\cdot\frac{b(1)}{1^s}.$$

More generally, the coefficient $c(n)$ in equation 1 is given by

$$c(n) = \sum_{d|n} a(d)\cdot b\left(\frac{n}{d}\right), \tag{2}$$

where $\sum_{d|n}$ means that the sum is over the divisors of n. This formula shows that we can find the multiplicative inverse of any nonzero series, so our operations make the set of all Dirichlet series into a field.

Perhaps the simplest Dirichlet series is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Formula (2) implies that if

$$\zeta(s) \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \sum_{n=1}^{\infty} \frac{c(n)}{n^s},$$

then

$$c(n) = \sum_{d|n} a(d). \tag{3}$$

Using this, we can find $1/\zeta(s)$. Define the numbers $\mu(n)$ to be the coefficients of the reciprocal of ζ, so that

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1.$$

Donning our Platonist hats, let's go on a hunt for μ. The definition of ζ and equation (3) imply that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

This gives a recursive formula for calculating μ:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ -\sum_{d\|n} \mu(d) & \text{otherwise,} \end{cases} \tag{5}$$

where $\sum_{d\|n}$ means that the sum is over the proper divisors of n. The idea is to pick away at what μ must be.

We know that $\mu(1) = 1$. If p is prime, then

$$\mu(p) = -\sum_{d\|p} \mu(d) = -\mu(1) = -1,$$

and

$$\mu(p^2) = -\sum_{d\|p^2} \mu(d) = -(\mu(1) + \mu(p)) = 0.$$

By induction, we see that $\mu(p^n) = 0$ for $n > 0$.

If p and q are primes, then

$$\mu(pq) = -\sum_{d\|pq} \mu(d) = -(\mu(1) + \mu(p) + \mu(q)) = 1.$$

Similarly, one can check all kinds of other special cases such as $n = pq^2$, $n = pqr$, and so on—these would be essential homework problems for a number theory class, and it leads to a result that is a nice exercise for students:

Fact 1. If p is a prime and $\gcd(p, n) = 1$, then $\mu(pn) = -\mu(n)$.

The proof is by induction (that's why it would be important to work out many special cases in problem sets):

$$\begin{aligned}
\mu(pn) &= -\sum_{d \| pn} \mu(d) \\
&= -\left(\sum_{d|n} \mu(d) + \sum_{d\|n} \mu(pd)\right) \\
&= -\left(\sum_{d|n} \mu(d) + -\sum_{d\|n} \mu(d)\right) \qquad \text{(by induction)} \\
&= -(0 + \mu(n)) \qquad \text{(from (4) and (5))} \\
&= -\mu(n).
\end{aligned}$$

We get a little machine going:

Fact 2. If p is prime and $\gcd(p, n) = 1$, then $\mu(p^2 n) = 0$.

Once again, assume it's true for all proper divisors of n (and check this for divisors of special forms, like primes or squares of primes). Then

$$\begin{aligned}
\mu(p^2 n) &= -\sum_{d \| p^2 n} \mu(d) \\
&= -\left(\sum_{d|n} \mu(d) + \sum_{d|n} \mu(pd) + \sum_{d\|n} \mu(p^2 d)\right) \\
&= -\left(\sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) + \sum_{d\|n} \mu(p^2 d)\right) \\
&= -\left(\sum_{d\|n} \mu(p^2 d)\right) \\
&= 0.
\end{aligned}$$

The following result can also be proved by induction.

Theorem. *If p is prime and* $\gcd(p, n) = 1$,

$$\mu(p^k n) = \begin{cases} -\mu(n) & \textit{if } k = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

From here, it's easy to get the usual formula for μ.

Once we determine that μ gives the coefficients for the reciprocal of ζ, we can establish many of the elementary identities of number theory by manipulating series. For example, Euler's ϕ function is defined by the rule

$\phi(n) =$ the number of positive integers less than n that are relatively prime to n.

Numerical experiments suggest that

$$\sum_{d|n} \phi(d) = n.$$

This is equivalent to

$$\zeta(s) \sum \frac{\phi(n)}{n^s} = \sum \frac{n}{n^s}.$$

And this is equivalent to

$$\sum \frac{\phi(n)}{n^s} = \sum \frac{\mu(n)}{n^s} \sum \frac{n}{n^s},$$

or

$$\phi(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right). \tag{6}$$

To establish this last identity, suppose first that n has only two prime factors: $n = p_1^{e_1} p_2^{e_2}$. Then to find $\phi(n)$, we start with n (the number of integers less than or equal to n), subtract n/p_1 (the number of integers less than or equal to n that are divisible by p_1), subtract from that n/p_2 (the number of integers less than or equal to n that are divisible by p_2), and then add back $n/p_1 p_2$, since the multiples of $p_1 p_2$ were subtracted twice.

If n has three prime factors, p_1, p_2, and p_3, a similar argument shows that

$$\phi(n) = n - \frac{n}{p_1} - \frac{n}{p_2} - \frac{n}{p_3} + \frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \frac{n}{p_2 p_3} - \frac{n}{p_1 p_2 p_3}.$$

This inclusion-exclusion argument generalizes: If

$$n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k},$$

then

$$\phi(n) = n \left(1 - \sum_i \frac{1}{p_i} + \sum_{i,j} \frac{1}{p_i p_j} - \sum_{i,j,l} \frac{1}{p_i p_j p_l} + \cdots + (-1)^k \frac{1}{p_1 \cdots p_k}\right).$$

One simplification of this equation is to factor the right-hand side, yielding the famous

$$\phi(n) = n \prod_i \left(1 - \frac{1}{p_i}\right).$$

But another simplification comes from the fact that the right-hand side can be written as:

$$\mu(1)n + \sum_i \mu(p_i)\frac{n}{p_i} + \sum_{i,j} \mu(p_i p_j)\frac{n}{p_i p_j} + \cdots + \mu(p_1 \cdots p_k)\frac{n}{p_1 \cdots p_k}. \qquad (7)$$

Since μ vanishes at factors of n that are divisible by the square of some p_i, expression (7) is none other than

$$\sum_{d|n} \mu(d)\left(\frac{n}{d}\right),$$

which gives identity (6), and that implies what we want.

More generally, the *Möbius Inversion Theorem* states that if

$$b(n) = \sum_{d|n} a(d),$$

then

$$a(n) = \sum_{d|n} \mu(d) b\left(\frac{n}{d}\right).$$

This follows from the equivalence

$$\zeta(s)\sum \frac{a(n)}{n^s} = \sum \frac{b(n)}{n^s} \iff \sum \frac{a(n)}{n^s} = \sum \frac{\mu(n)}{n^s}\sum \frac{b(n)}{n^s}.$$

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