accounting for Byer's observation that

$$r(\vec{x}(n), \vec{y}_m(k, s)) = 0,$$

where m is the middle index.

Byer concludes by observing that for a fixed vector $\vec{x}$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$,

$$\lim_{y_n \to \infty} r(\vec{x}, \vec{y}) = r(\vec{x}, \vec{y}_n(0, 1)).$$

Byer proves this fact by algebraically manipulating (1). Taking a linear algebra perspective, we can place this in a more general context and provide better intuition. Fix vectors $\vec{x}$, $\vec{y}$, and $\vec{u}$. Then use of (6) yields

$$\begin{aligned}
\lim_{s \to \infty} r(\vec{x}, \vec{y} + s\vec{u}) &= \lim_{s \to \infty} \frac{\langle \widehat{x}, \vec{y} + s\vec{u} \rangle}{\|\widehat{x}\| \, \|\widehat{y} + s\widehat{u}\|} \\
&= \lim_{s \to \infty} \frac{\langle \widehat{x}, \frac{1}{s}\vec{y} + \vec{u} \rangle}{\|\widehat{x}\| \, \|\frac{1}{s}\widehat{y} + \widehat{u}\|} \\
&= \frac{\langle \widehat{x}, \vec{u} \rangle}{\|\widehat{x}\| \, \|\widehat{u}\|} \\
&= r(\vec{x}, \vec{u}).
\end{aligned}$$

We have seen how ideas from linear algebra can give insight into the sample correlation coefficient. They can also provide intuition in the study of correlation and covariance of distributions. While none of these ideas are new or deep, they should enable the reader to appreciate the explanatory power of a linear algebra approach in a statistical context. Sadly, while some linear algebra texts like [**2**] include applications to statistics, available statistics texts, perhaps in an effort to reduce prerequisites, do not seem to relate concepts in statistics to ideas from linear algebra. Even [**3**], which uses ideas from matrix theory in the context of analysis of variance and which proves and names a version of the Cauchy-Schwartz inequality in the context of the covariance of distributions, does not point out the relationship between the inner product and covariance. This is an unfortunate loss of a helpful tool for understanding covariance and correlation. It is always a good idea to find ways to reference, use, and reinforce ideas studied in other courses.

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Pythagoras by the Cross Ratio

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There are numerous proofs of the *Pythagorean theorem*. These include the more than three hundred and fifty demonstrations found in [**2**] alone. However, we have not encountered a proof that uses the cross ratio. The objective of this note to show how the cross ratio as applied by O. Shisha to prove Ptolemy's Theorem in [**3**] may also be utilized to demonstrate the Pythagorean theorem. Our proof represents a joint effort following the introduction of the cross ratio in a complex analysis class taught by the second author and taken by the first author.

Following [**1**], we define the *cross ratio* of four distinct complex numbers, z_1, z_2, z_3, and z_4 as

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}. \qquad (1)$$

Let ABC be any right triangle in the complex plane and let D be the point lying on the circle that passes through A, B, and C such that $ABCD$ forms a rectangle. Then A, B, C, and D lie on the circle in this order.

Let

$$\alpha = |AD| = |BC|, \quad \beta = |AB| = |CD|, \quad \text{and} \quad \gamma = |AC| = |BD|.$$

Now, let w be the linear fractional transformation that maps the above circle onto the real axis according to

$$w(B) = 1, w(C) = 0, \quad \text{and} \quad w(D) = \infty.$$

Hence, $w(A) > 1$.

Let $w_i = w(z_i)$ for $1 \leq i \leq 4$; then by (1) and the fact that

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)},$$

it follows that

$$\frac{\alpha^2}{\gamma^2} = \frac{|AD| \cdot |BC|}{|AC| \cdot |BD|} = |(A, D, B, C)| = |(f(A), \infty, 1, 0)| = \frac{1}{f(A)}.$$

Also,

$$\frac{\beta^2}{\gamma^2} = \frac{|AB| \cdot |DC|}{|AC| \cdot |BD|} = |(A, B, D, C)| = |(f(A), 1, \infty, 0)| = \frac{f(A) - 1}{f(A)}.$$

Thus,

$$\frac{\alpha^2}{\gamma^2} + \frac{\beta^2}{\gamma^2} = \frac{1}{f(A)} + \frac{f(A) - 1}{f(A)} = 1.$$

Therefore,

$$\alpha^2 + \beta^2 = \gamma^2.$$

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Parity and Primality of Catalan Numbers

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Catalan numbers, like Fibonacci and Lucas numbers, appear in a variety of situations, including the enumeration of triangulations of convex polygons, well-formed sequences of parentheses, binary trees, and the ballot problem [**1**]–[**5**]. Like the other families, Catalan numbers are a great source of pleasure, and are excellent candidates for exploration, experimentation, and conjecturing.

They are named after the Belgian mathematician Eugene Catalan (1814–1894), who discovered them in his study of well-formed sequences of parentheses. However, Leonhard Euler (1707–1783) had found them fifty years earlier while counting the number of triangulations of convex polygons [**3**]. But the credit for the earliest known discovery goes to the Chinese mathematician Antu Ming (ca. 1692–1763), who was aware of them as early as 1730 [**6**].

In 1759 the German mathematician and physicist Johann Andreas von Segner (1707–1777), a contemporary of Euler, found that the number C_n of triangulations of a convex polygon satisfies the recursive formula

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0, \tag{1}$$

where $C_0 = 1$ [**3, 5**]. The numbers C_n are now called *Catalan numbers*. It follows from (1) that $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, and so on.

Using generating functions and Segner's formula, an explicit formula for C_n can be developed [**5**]:

$$\begin{aligned} C_n &= \frac{(2n)!}{(n+1)!n!} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned} \tag{2}$$

Consequently, C_n can be extracted from Pascal's triangle by dividing the central binomial coefficient $\binom{2n}{n}$ in row $2n$ by $n + 1$.

In this note, we identify Catalan numbers that are odd, and those that are prime. In Table 1, which gives the first eighteen Catalan numbers, those that are odd are marked with asterisks and those that are prime with daggers.

Parity of Catalan Numbers. It follows from the table that when $n \leq 17$, C_n is odd for $n = 0, 1, 3, 7$, and 15, all of which are of the form $2^m - 1$. When $m > 0$, such numbers are known as Mersenne numbers.

Theorem. *For $n > 0$, C_n is odd if and only if n is a Mersenne number.*

