References

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Tennis (and Volleyball) Without Geometric Series

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In a recent issue of this *Journal*, Sandefur [**2**] presented an analysis of the probability of winning a deuce game in tennis (that is, any game that is tied at three points or more). Because a player must win by at least two points, this analysis led naturally to a geometric series representation of the probability that the server wins the game.

There is an alternative approach to the problem that does not rely on geometric series. Suppose that A represents a point the server wins and b represents a point that the server loses. Given that the game is at deuce, there are only four possibilities for the next two points: AA, Ab, bA, and bb. If the server wins service with probability p, these four possibilities occur with respective probabilities of p^2, $p(1-p)$, $(1-p)p$, and $(1-p)^2$. In the first case, the server wins the game; in the last case, the server loses the game; and in the other two cases, the score returns to deuce. In tennis, the scores after a 3–3 tie are frequently referred to as 'deuce' if the score is tied, 'adin' (for "advantage in") if the server is up by one point, and 'adout' if the server is down by one point, without reference to the actual score.

Let α be the probability of the server winning a game when serving from deuce. If all serves are independent,

$$\alpha = p^2 \cdot 1 + 2p(1-p) \cdot \alpha + (1-p)^2 \cdot 0.$$

Hence,

$$\alpha = \frac{p^2}{1 - 2p(1-p)}, \tag{1}$$

as obtained in [**2**]. Note that the actual score at deuce is irrelevant; given that the player is serving a game from any deuce score, the probability of subsequently winning that game is α. To obtain the overall probability that the server wins a game from the beginning, we need to multiply the probability of reaching deuce the first time by α and add this to the probability of winning without reaching deuce (as is done in [**2**]).

Of course, this approach is not all that different from the usual method of finding the sum of a geometric series. However, it has the advantage of being applicable to other more complicated situations. We present three specific examples here: the first is for determining the expected length of a game of tennis; the other two are for the probability of winning and the expected length of a game in volleyball (due to having slightly different rules, these are more complicated than for tennis).

Expected length of a tennis game The length of a sporting event may be of some interest to its organizers, as well as to broadcasters covering it. For tennis, we use the number of serves as a measure of the length of a match. To begin our analysis, we let L be the number of additional serves in a game of tennis that has reached deuce. As before, we let p denote the probability that the server wins each point, and let $q = 1 - p$. Since the possibilities for the next two points are the same as listed above, the probability that the game ends after two more serves is $p^2 + q^2$, while the probability it goes beyond two serves is $2pq$. So, $E(L)$, the expected value of L, satisfies

$$E(L) = (p^2 + q^2) \cdot 2 + 2pq \cdot [2 + E(L)].$$

Solving this equation produces

$$E(L) = \frac{2}{1 - 2pq}. \tag{2}$$

As would be expected, if p is near 0 or near 1, then $E(L)$ is only slightly more than 2. On the other hand, if $p = 1/2$, that is, the two opponents are evenly matched, then on average it will take four serves from deuce to end the game. (It is straightforward to show that $E(L)$ has an absolute maximum at $p = 1/2$ on [0, 1].)

It is also interesting to note that, $E(L)$, unlike α, is symmetric in p and q. Obviously, the expected length of the match will be the same whether it is viewed from the perspective of the server or the receiver, where as in general the probability of winning will not be the same.

Winning in volleyball In volleyball, as opposed to tennis, the winner of any point during a game gets to serve for the next point. Under the current point per play scoring system (also called "quick" or "rally" scoring), a game is played to 30 points (or 15, in a fifth, and deciding, game) with the provision that you must win by 2 points. Thus, like tennis, volleyball has "deuce" games, the difference being that either team may be serving any of the subsequent points. (We note in passing that an analysis of several possible volleyball scoring systems may be found in [**1**].)

Let **A** and **B** be two volleyball teams, and suppose they win their serves with probability p and r, respectively. Further, let $q = 1 - p$ and $s = 1 - r$. Let A denote a point **A** serves and wins, while b denotes a point **A** serves and loses (as earlier). Similarly, let B denote a point **B** serves and wins, while a denotes a point **B** serves and loses. If A begins service at deuce, the possibilities for the next two points are, AA, Ab, ba, and bB. As before, **A** wins in the first case and loses in the last. The middle two cases return to deuce, with the difference being that **B** serves the next point in case two and **A** serves the next point in case three. If **B** begins service at deuce, the situation is reversed. The possibilities for the next two points are BB, Ba, ab, and aA. **B** wins in the first case and loses in the fourth; the middle two cases return to deuce with **A** serving the next point in case two and **B** serving the next point in case three.

Now, let α be the probability **A** wins a deuce game when serving the initial point from deuce, and let β be the corresponding probability for **B**. It follows that

$$\alpha = p^2 \cdot 1 + pq \cdot (1 - \beta) + qs \cdot \alpha + qr \cdot 0$$

and

$$\beta = r^2 \cdot 1 + rs \cdot (1 - \alpha) + sq \cdot \alpha + sp \cdot 0.$$

By straightforward algebra, we find that

$$\alpha = \frac{p^2}{1 - qs(1 + p + r)} \tag{3}$$

and

$$\beta = \frac{r^2}{1 - qs(1 + r + p)}.$$

(The formula for β can also be obtained by interchanging p with r and q with s in the formula for α.) In the case where who wins a point does not depend on who serves, then $p = s$ and $q = r$ and equation (3) reduces to equation (1), and $\beta = 1 - \alpha$, as would be expected.

Expected length of a volleyball game Suppose we now let M be the number of additional serves from deuce in a volleyball game when **A** begins serving, and let N be the corresponding value when **B** begins serving. In either case, the possibilities for the next two serves and their probabilities are the same as in the preceding section. Thus, $E(M)$, the expected value of M, satisfies

$$E(M) = 2 + pq\, E(N) + qs\, E(M),$$

and, similarly,

$$E(N) = 2 + rs\, E(M) + qs\, E(N).$$

A little algebra produces

$$E(M) = \frac{2[1 + q(p - s)]}{[(1 - qs)^2 - pqrs]} = \frac{2[1 + q(p - s)]}{1 - qs(2 - qs + pr)}. \tag{4}$$

As before, we can solve for $E(N)$ directly, or we can obtain it from (4) by interchanging p with r and q with s. If the probability that **A** wins each point does not depend on which team begins serving, then $p = s$ and $q = r$, and (4) reduces to (2).

For two teams of equal strength (in both serving and receiving), $p = r$ and $q = s$, and (4) reduces to

$$E(M) = \frac{2[1 + q(p - q)]}{1 - q^2(2 - q^2 + p^2)}.$$

If, in addition, $p = q = 1/2$, then $E(M) = E(N) = 4$, as was the case in tennis. However, this is no longer the maximum value for $E(M)$. In fact, $E(M)$ can take on any positive value greater than 2. If p is close to 1, then $E(M)$ will be slightly larger than 2. If p is close to 0 (so that q is close to 1), then $E(M)$ will be very large. In this case, nearly every game will reach deuce and continue long after deuce is reached. This reflects the fact that in volleyball, to win by two you must win at least one of your own serves.

In current intercollegiate and international volleyball, both p and r are generally around 0.3, so that $E(M) \approx 20/3$. That is, a typical deuce game between roughly equal opponents should take another 6 or 7 serves to determine the winner.

Although the examples in this note are extensions of the sports context discussed by Sandefur [**2**], this recursive method can be applied to a much wider class of problems. In my experience this method can be applied to most situations where geometric series are typically used.

References

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2. J. Sandefur, A geometric series from tennis, *College Math. J.*, **36** (2005) 224–226.