Thus

$$\begin{aligned} S(a) &= |b-a| + |c-a| \\ &> |f-b| + |f-a|/2 + |f-c| + |f-a|/2 \\ &= S(f). \end{aligned}$$

Similarly $S(b) > S(f)$, $S(c) > S(f)$. This shows that S does not attain its minimum at any vertex. It follows that S must attain its minimum at some interior point R; and then, by the Proposition, $DS(r) = 0$. But F is the unique point for which $DS(r) = 0$; so $R = F$.

Case 2. Suppose, alternatively, that one of the angles A, B, C is not less than 120°. Then there can be no point R such that RA, RB, RC make equal angles with each other; so S cannot attain its minimum at an interior point, and must therefore attain its minimum at a vertex. Since the longest side of the triangle is the one opposite the largest angle, we see that S attains its global minimum at the vertex of the largest angle.

We have demonstrated the following.

THEOREM. *If all the angles of the triangle ABC are less than* 120° *then the Fermat point F is the point such that FA, FB, FC meet at* 120°; *otherwise it is the vertex of the largest angle.*

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REFERENCES

1. R. Courant and H. Robbins, *What is Mathematics*?, Oxford University Press, Oxford, UK, 1941.
2. Mowaffaq Hajja, An advanced calculus approach to finding the Fermat point, this MAGAZINE Vol. 67, 1994, 29–34.
3. J. Pottage, *Geometric Investigations*, Addison-Wesley, Reading, MA, 1983.

Determinants of the Tournaments

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In a round-robin tournament with n players, each player plays every other player in a game where ties are not possible. The results of the tournament can be summarized by an n by n *tournament matrix* A whose (i, j) entry is 1 if i beat $j, -1$ if j beat i, and 0 if i equals j. The matrix below represents a tournament where, for example, player 1 beat players 2 and 4, but lost to players 3 and 5. The authors confess that the paper was motivated by *word play* with the hope of determining that determinants and tournaments have more in common than their names suggest. We discovered that in fact the concepts are almost *independent*, but do provide an opportunity to illustrate several powerful types of determinant arguments.

$$\begin{bmatrix} 0 & 1 & -1 & 1 & -1 \\ -1 & 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}$$

Proposition 1. *Let A be the matrix representation of a tournament with n players. The determinant of A is zero if, and only if, n is odd.*

Proof. Any tournament matrix A is necessarily skew-symmetric, i.e., $A = -A^T$. Therefore, $\det(A) = \det(-A^T) = (-1)^n \det(A^T) = (-1)^n \det(A)$. When n is odd, $\det(A) = -\det(A)$ and must therefore be zero.

For the case where n is even, recall that to compute the determinant, we can determine it by summing products of the terms in it according to the formula:

$$\det(A) = \sum_{p \in S_n} \operatorname{sign}(p) a_{1,p(1)} a_{2,p(2)} \cdots a_{n,p(n)}$$

where S_n is the set of all permutations on n elements. We shall show that this determinant is odd, and hence nonzero. Since each $a_{i,j}$ is $0, 1$, or -1, so is the product $\operatorname{sign}(p) a_{1,p(1)} a_{2,p(2)} \cdots a_{n,p(n)}$. If $p(i) = i$ for some i, then $a_{i,p(i)}$ is 0, and hence $\operatorname{sign}(p) a_{1,p(1)} a_{2,p(2)} \cdots a_{n,p(n)}$ is 0. So we only need to take the sum over all permutations that do not map any element to itself, since all other permutations contribute zero to the sum. Since for each such permutation p, $\operatorname{sign}(p) a_{1,p(1)} a_{2,p(2)} \cdots a_{n,p(n)}$ is 1 or -1, we can calculate $\det(A)$ modulo 2, simply by counting the number of *derangements*, permutations that do not map any element to itself.

By the principle of inclusion-exclusion, there are

$$\sum_{i=0}^{n} (-1)^i (n-i)! \binom{n}{i}$$

derangements. Since $(n-i)!$ is even for $i \leq n-2$, the previous summation has the same parity as

$$(-1)^{n-1} 1! \binom{n}{n-1} + (-1)^n 0! \binom{n}{n} = -n - 1$$

which is odd. Thus, $\det(A)$ is nonzero.

Here is another simple proof for the case when n is even. Since $\det(A) \bmod 2$ is unaffected by changing (-1)'s into 1's, it suffices to compute the parity of the determinant of the matrix

$$J - I = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

where I is the n by n identity matrix, and J is the n by n matrix consisting entirely of 1's.

For any matrix C, we have

$$C\vec{x} = \lambda\vec{x} \Rightarrow (C - I)\vec{x} = (\lambda - 1)\vec{x},$$

so the eigenvalues of $J - I$ are all one less than the eigenvalues of J.

The rank of J is 1, so 0 is an eigenvalue with multiplicity $n - 1$. Since n is also an eigenvalue for J (with eigenvector $[1, 1, \ldots, 1]^T$), its multiplicity must be 1. So $J - I$ has the eigenvalue -1 with multiplicity $n - 1$ and the eigenvalue $n - 1$ with multiplicity 1. Hence the determinant of $J - I$ equals the product of its eigenvalues, namely $(-1)^{(n-1)}(n - 1)$, which is odd.

Yet another way to compute $\det(J - I)$ is by performing elementary row and column operations that do not affect the determinant. (This argument can be applied to any square matrix with one number on the main diagonal and another number everywhere else. See for instance, [1].) Adding every row of $J - I$ (except the first) to the first row gives us the matrix

$$\begin{bmatrix} n-1 & n-1 & n-1 & \cdots & n-1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}.$$

After subtracting the first column of this matrix from all the other columns we obtain the lower triangular matrix below with determinant $(-1)^{(n-1)}(n - 1)$

$$\begin{bmatrix} n-1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}.$$

In fact, a little more can be said about the "zeroness" of $\det(A)$.

Proposition 2. *The nullspace of a tournament matrix A has dimension zero if n is even, and dimension one if n is odd.*

Proof. If n is even, then A is nonsingular and the proposition follows. For odd n, let $\vec{a}_1, \ldots, \vec{a}_n$ be the columns of A. Let C be the $(n - 1) \times (n - 1)$ matrix which results from deleting the last row and column from A. This matrix then corresponds to some tournament on $n - 1$ vertices. Hence, since $n - 1$ is even, the above result implies that C is nonsingular. The columns of this matrix are therefore linearly independent. It follows that the vectors $\vec{a}_1 \cdots \vec{a}_{n-1}$ are linearly independent, since they are the columns of C with an additional component. So the rank of A is at least $n - 1$. If A had rank n, it would be nonsingular, which we know to be false. So the rank of A is $n - 1$, and the dimension of its nullspace is therefore 1.

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REFERENCE

Victor Bryant, *Aspects of Combinatorics*, Cambridge University Press, Cambridge, UK, 1993, p. 182.