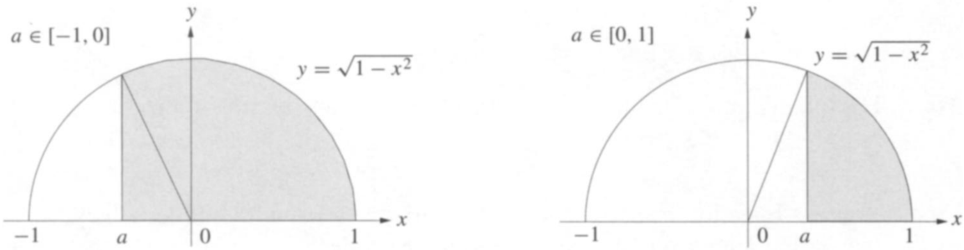
Proof Without Words: Look Ma, No Substitution!

$$\int_a^1 \sqrt{1-x^2}dx = \frac{\cos^{-1} a}{2} - \frac{a\sqrt{1-a^2}}{2}, \; a \in [-1, 1].$$



$$\int_a^1 \sqrt{1-x^2}dx = \frac{\cos^{-1} a}{2} + \frac{(-a)\sqrt{1-a^2}}{2}$$

$$\int_a^1 \sqrt{1-x^2}dx = \frac{\cos^{-1} a}{2} - \frac{a\sqrt{1-a^2}}{2}$$

—MARC CHAMBERLAND
GRINNELL COLLEGE
GRINNELL, IA 50112

Permutations and Coin-Tossing Sequences

DAVID CALLAN
Department of Statistics
University of Wisconsin
Madison, WI 53706-1693

"The answer can be restated in quite a striking form, like this

THEOREM 3.7.1. *Let a positive integer n be fixed. The probabilities of the following two events are equal:*

(a) *a permutation is chosen at random from among those of n letters, and it has an even number of cycles, all of whose lengths are odd*

(b) *a coin is tossed n times and exactly n/2 heads occur.*"

This quote is from Herbert Wilf's delightful book *generatingfunctionology* [**1**, p. 75]. It occurs in the chapter on the exponential formula, a powerful technique for counting labelled structures formed from "connected" components. Such structures include various types of graphs, permutations (formed from cycles), and partitions (a union of blocks). Applied to permutations on n letters comprising an even number of cycles all of whose lengths are odd, the exponential formula shows that there are $\binom{n}{n/2}\frac{n!}{2^n}$ of them. Thus the probability in (a) is $\binom{n}{n/2}\frac{1}{2^n}$; this is obviously also the probability in (b), and the quoted theorem follows. The purpose of the present note is to offer a combinatorial explanation of this "striking" result.

Both probabilities are 0 if n is odd; so assume n is even. Now let $\mathcal{A}_n$ denote the permutations in (a) and let $\mathcal{B}_n$ denote the coin-tossing sequences in (b). Thus $\mathcal{A}_n$ is the set of permutations on $[n] = \{1, 2, \ldots, n\}$ all of whose cycles are of odd length (their number has to be even since n is even). We can take $\mathcal{B}_n$ to be the set of 0-1 sequences comprising m 1s and m 0s (where $m = n/2$). Let $\mathcal{S}_n$ denote the set of all $n!$ permutations on $[n]$ and $\mathcal{T}_n$ the set of all 2^n 0-1 sequences of length n. Then the Theorem asserts

$$\frac{|\mathcal{A}_n|}{|\mathcal{S}_n|} = \frac{|\mathcal{B}_n|}{|\mathcal{T}_n|} \tag{1}$$

We will "explain" this coincidence of probabilities by constructing a bijection

$$\mathcal{A}_n \times \mathcal{T}_n \longrightarrow \mathcal{B}_n \times \mathcal{S}_n \tag{2}$$

First, we give a bijection from $\mathcal{A}_n$—the permutations on $[n]$ with odd-length cycles—to $\mathcal{C}_n$, the permutations on $[n]$ with even-length cycles. To do so, say a permutation is in *standard cycle form* when its cycles are arranged so that the largest element in each cycle occurs first, and these first elements are in increasing order. Thus $(5, 3, 1)(8, 2, 6)(9)(12, 10, 4, 11, 7)$ is in standard cycle form. Given $\pi \in \mathcal{A}_n$ in standard form, move the last element of the cycles in the 1st, 3rd, 5th, ... positions to the end, respectively, of the cycles in 2nd, 4th, 6th, ... positions. Thus $(5, 3, 1)(8, 2, 6)(9)(12, 10, 4, 11, 7) \rightarrow (5, 3)(8, 2, 6, 1)(12, 10, 4, 11, 7, 9)$. This is the desired bijection. Note the resulting permutation is in $\mathcal{C}_n$ and is again in standard form. To reverse the mapping, delete the last element of the last cycle. Place it at the end of the preceding cycle provided this gives a legitimate cycle (largest element first); otherwise create a new 1-cycle consisting of this element alone. Proceed similarly so that each of the original even-length cycles either has its last element deleted or acquires a new last element.

Second, we give a bijection between $\mathcal{C}_n$—the permutations on $[n]$ with even-length cycles—and $\mathcal{D}_n \times \mathcal{D}_n$ where $\mathcal{D}_n$ is the permutations on $[n]$ with all cycles of length 2 (transpositions). An element $\pi \in \mathcal{D}_n$ can be represented as a graph on the vertices $[n]$ in which each vertex has degree 1 (two vertices are joined precisely when they occur in the same transposition). Thus $(\pi_1, \pi_2) \in \mathcal{D}_n \times \mathcal{D}_n$ is a pair of such graphs as illustrated in FIGURE 1 (with $n = 6$; solid edges for the first graph, dotted edges for the second).

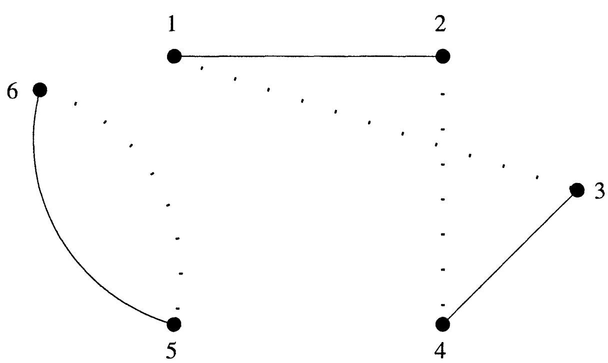


Figure 1

The union of the edge sets is a collection of (unoriented) cycles of even length ≥ 2 with alternating solid and dotted edges. This yields even-length cycles on $[n]$ just as we want except that we must orient the cycles of length ≥ 4 in one of two possible ways. But there are two possible patterns for the solid and dotted edges in such a cycle, so all is well. For definiteness, orient each cycle in the direction of, say, the solid edge emanating from its smallest vertex.

These bijections show that the left side of (2) is $\approx \mathcal{D}_n \times \mathcal{D}_n \times \mathcal{T}_n$ and hence $\approx \mathcal{D}_{2m} \times \mathcal{D}_{2m} \times \mathcal{T}_m \times \mathcal{T}_m$ (recall $n = 2m$). Turning to the right side of (2), we observe that there is a bijection $\mathcal{S}_{2m} \longrightarrow \mathcal{B}_{2m} \times \mathcal{S}_m \times \mathcal{S}_m$. Given $\pi \in \mathcal{S}_{2m}$, the *locations* of $1, 2, \dots, m$ in π give an element of $\mathcal{B}_{2m}$, the *order* of $1, 2, \dots, m$ in π gives an element of $\mathcal{S}_m$, and the order of $m+1, m+2, \dots, 2m$ gives another. For example, $\left(\begin{smallmatrix}1&2&3&4&5&6\\4&6&3&5&1&2\end{smallmatrix}\right)$ yields $(0\,0\,1\,0\,1\,1) \times \left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right) \times \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right) \in \mathcal{B}_{2m} \times \mathcal{S}_m \times \mathcal{S}_m$ where in the last permutation 1 3 2 is the rank ordering of 4 6 5. Hence the right side of (2) is $\approx \mathcal{B}_{2m} \times \mathcal{B}_{2m} \times \mathcal{S}_m \times \mathcal{S}_m$.

Thus we can identify a "square root" of each side of (2) and it now suffices to exhibit a bijection

$$\mathcal{D}_{2m}\left(\begin{smallmatrix}\text{products of}\\ \text{disjoint 2-cycles}\end{smallmatrix}\right) \times \mathcal{T}_m\left(\begin{smallmatrix}\text{unrestricted}\\ \text{0-1 sequences}\end{smallmatrix}\right) \longrightarrow \mathcal{B}_{2m}\left(\begin{smallmatrix}\text{sequences of}\\ m\text{ 0s, } m\text{ 1s}\end{smallmatrix}\right) \times \mathcal{S}_m\left(\begin{smallmatrix}\text{unrestricted}\\ \text{permutations}\end{smallmatrix}\right) \quad (3)$$

This is quite easy: given $(\pi, \epsilon) \in \mathcal{D}_{2m} \times \mathcal{T}_m$, start with π in standard cycle form. Reverse the transpositions located in those positions where ϵ has a 1, and then arrange the transpositions in the order of their first elements. For example, with $m = 4$, $\pi = (3,2)(5,1)(6,4)(8,7)$ (in standard cycle form), and $\epsilon = (1,0,1,1)$, $(\pi,\epsilon) \to (2,3)(5,1)(4,6)(7,8) \to (2,3)(4,6)(5,1)(7,8)$. The first elements of the final product of transpositions form an m-element subset of $[2m]$ determining an element of $\mathcal{B}_{2m}$, while the rank ordering of the second elements is a permutation in $\mathcal{S}_m$. The example yields $\{2,4,5,7\} \to (0,1,0,1,1,0,1,0) \in \mathcal{B}_8$ and $(3,6,1,8) \to (2,3,1,4) \in \mathcal{S}_4$. The original pair (π, ϵ) can be uniquely retrieved, and the bijection (3) is established.

REFERENCES

1. Herbert S. Wilf, *generatingfunctionology*, Academic Press, New York, NY, 1990.

An Easy Solution to *Mini Lights Out*

JENNIE MISSIGMAN
RICHARD WEIDA
Lycoming College
Williamsport, PA 17701

In this MAGAZINE [**1**], Anderson and Feil demonstrated how to use linear algebra to solve the game *Lights Out*, which consists of a 5×5 array of lighted buttons; each light may be on or off. Pushing any button changes the on/off state of that light as well as the states of all its vertical and horizontal neighbors. Given a particular configuration of lights which are turned on, the object of the game is to turn out all the lights. While the computation of the solution in [**1**] is relatively straightforward, it certainly cannot be accomplished by hand in a reasonable amount of time. Analysis of a somewhat similar 3×3 game, *Merlin's Magic Square*, can be found in [**2, 3**].

Tiger Electronics has recently released a new version of the game, called *Mini Lights Out*. This consists of a 4×4 array of lighted buttons, but this time, unlike the original 5×5 version, "on" a torus. That is, the uppermost and lowermost rows are considered neighbors and likewise the leftmost and rightmost columns are considered neighbors.