Fetching Water with Least Residues

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A classic problem has a person, we'll call her Jill, going to a well with two jugs, one holding three pints and the other five, and she is to return with precisely four pints of water. A version of this problem appears in the movie *Die Hard with a Vengeance*, in which a solution is needed to prevent a bomb from detonating (but one step in the solution was left out). The script and a discussion of the movie are available online (see [**5**] and [**6**]). Peterson [**2**] summarizes some of the recent related generalizations and notes that the problem dates back at least to the thirteenth century.

For an illustration, we vary the problem a bit, asking for two pints rather than four, but using the same size jugs, a 3-pint jug J_3 and a 5-pint jug J_5. Here are two solutions to this problem:

(1) Fill J_5 and from it fill J_3, leaving 2 pints in J_5.

(2) Fill J_3 and pour this into J_5. Refill J_3 and from it fill J_5, leaving 1 pint in J_3. After emptying J_5, pour the 1 pint into J_5. Refill J_3 again and pour this into J_5, then fill J_3 again and from it fill J_5, leaving the desired 2 pints in J_3.

Clearly if efficiency is desired, as it was in the movie, then the first solution is the one to use.

The general two-jug problem. In the general problem, Jill goes to the well with an m-pint jug J_m and an n-pint jug J_n, with $m < n$, and she is to bring back exactly p pints of water. The first problem is to determine whether this is possible. If it is, the second problem is to find the most efficient way to do it. The first problem has been solved many times in many ways. For example, Pfaff and Tran [**3**] used the Euclidean Algorithm in their solution. In this note we use properties of least residues to show that there are always exactly two distinct sequences of pourings that solve the problem. Furthermore, the one with the fewer "stages" can be determined from a linear congruence, without listing the details of the sequences.

If the greatest common divisor of m and n is $g > 1$, then when there is a solution, g must also divide p since p is a linear combination of m and n. We can thus choose g pints as the unit of volume and solve the problem accordingly. Hence, in the remainder of this note, we assume that m and n are relatively prime. Since the combined capacity of the jugs is $m + n$, $0 \le p \le m + n$. It suffices to solve the problem for $p < m$ since multiples of m can always be added to a solution as long as we stay below $m + n$. Our solution is based on some applications of least residues.

Solution using least residues. The least residue of an integer k (mod m) is that integer r, $0 \le r \le m - 1$, for which $k = qm + r$ for some integer q. We denote that residue of k (mod m) by $L_m(k)$. Three elementary properties of this function that we will use are the following: For integers i and j,

(a) $L_m(i + mj) = L_m(i)$,

(b) $L_m(-ij) = L_m((m - i)j)$, and

(c) $L_m(i + L_m(j)) = L_m(i + j)$.

Results (a) and (b) follow immediately from the definition of least residue, while (c) follows from (a) if $L_m(j)$ is replaced by $j - qm$.

Assume that at some stage Jill has x pints, with $x < m$ in one of the jugs. She then has two options for going to the next stage:

Option A: With the x pints in J_m, she can fill J_n and from it fill J_m, leaving $n - (m - x)$ pints in J_n. By emptying multiples of m from J_n, she can reduce the amount to $L_m(n - m + x)$, that is, to $L_m(n + x)$. We denote this amount by $R_A(x)$.

Option B: With the x pints in J_n, she can repeatedly use J_m to eventually fill J_n, when there will be $tm - (n - x)$ pints in J_m for some positive integer t. This amount can be seen to equal $L_m(x - n)$, which we denote by $R_B(x)$.

Note that nothing is gained by mixing options since they cancel one another out. Thus, a solution will consist of a sequence of option A actions or a sequence of option B actions. (Grossman [**1**] also showed that there are exactly two distinct pouring sequences.)

When Jill arrives at the well, she has no water, and thus the initial value of x is 0. Using option A gives $R_A(0) = L_m(n)$. Applying option A a second time, with $x = L_m(n)$ and using the least residue result (c), gives $R_A(L_m(n)) = L_m(L_m(n) + n) = L_m(2n)$. Similar calculations using option A for $m - 1$ stages give the sequence of amounts

$$0, L_m(n), L_m(2n), \ldots, L_m((m - 1)n). \tag{1}$$

The corresponding result using $m - 1$ stages of option B gives the sequence

$$0, L_m(-n), L_m(-2n), \ldots, L_m(-(m - 1)n). \tag{2}$$

The m integers in (1) are nonnegative and less than m, and it is not difficult to see that they therefore form a permutation of $0, 1, 2, \ldots, m - 1$. Applying the least residue result (b) gives $L_m(-in) = L_m((m - i)n)$. Hence, if Jill uses repeated application of either option, then she can return with any integer number of pints of water from 0 through $m - 1$. By adding multiples of m she can return with any integer amount up to $m + n$.

Minimizing the number of stages. If each application of option A or B is called a *stage*, then we want to determine which option requires fewer stages. One way to determine this is to calculate the amounts in (1) and (2) and choose the option with fewer stages. For example, if $m = 9$ and $n = 34$, then, as the reader can readily check, the option A sequence is 0, 7, 5, 3, 1, 8, 6, 4, 2, from which it follows that the option B sequence is 0, 2, 4, 6, 8, 1, 3, 5, 7. Hence, if Jill is to return with 4 pints, then 7 stages are required using option A but only 2 using B.

The better option can also be determined without calculating each of the least residues. Let $s_A(p)$ denote the number of stages needed to obtain p pints of water using A. This number can be found by solving the linear congruence $s_A(p)n \equiv p \pmod{m}$. The required number of stages using option B is denoted $s_B(p)$, and equals $m - s_A(p)$. Thus she should choose the option corresponding to the smaller of $s_A(p)$ and $s_B(p)$, or either if they are equal. The congruence can be simplified by replacing n with $L_m(n)$ and replacing p with $p + tm$, for suitable t. For example, if $m = 9$, $n = 34$, and $p = 4$, we must solve $34s_A(4) \equiv 4 \pmod{9}$, $0 \le s_A(4) < 9$, or equivalently, $7s_A(4) \equiv 4 \pmod{9}$. Thus (as noted above), $s_A(4) = 7$ and $s_B(4) = 2$, so Jill should use option B twice to return with 4 pints. For the first application, she pours 27 pints into J_{34} leaving room for 7 pints. She then fills J_9 again and tops off J_{34} leaving 2 pints in J_9. She then applies option B a second time, pouring the 2 pints into J_{34} (after emptying it), leaving room for 32 pints, and again tops off J_{34}, leaving 4 pints in J_9.The reader is invited to solve the appropriate linear congruence to determine which option results in fewer stages if Jill has the 3-pint and 5-pint jugs and is to return with 1 pint.

Note that some of the stages require more pourings than others, and thus one can search for the option with the fewest pourings rather than the minimum number of stages. Another possible extension is to apply this method to problems with more than two jugs, something that has been studied by a number of authors; see Tweedie [**4**].

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Teaching Tip: Partial Fractions or No Partial Fractions?

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The standard method of integrating a rational function is to use partial fraction decomposition. However, if the resulting fractions involve a large number of coefficients, the algebra gets extremely tedious. In some such cases, one can use the technique of "factor creation" to evaluate the integral more readily. For example,

$$\begin{aligned}\int \frac{64}{x^6(x+2)}\,dx &= \int \frac{x^6+64-x^6}{x^6(x+2)}\,dx \\ &= \int \frac{1}{x+2}\,dx + \int \frac{(8-x^3)(x-2x+x^2)}{x^6}\,dx \\ &= \int \frac{1}{x+2}\,dx + \int \frac{8x^2-16x-4x^3+2x^4-x^5+32}{x^6}\,dx \\ &= \ln|x+2| - \ln|x| - \frac{2}{x} + \frac{2}{x^2} - \frac{8}{3x^3} + \frac{4}{x^4} - \frac{32}{5x^5} + C.\end{aligned}$$

Interested readers may find it instructive to apply the method to the following:

(a) $\displaystyle\int \frac{16}{x^8(x^2+2)}\,dx$ *Ans.* $\frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}x}{2}\right) + \frac{1}{x} - \frac{2}{3x^3} + \frac{4}{5x^5} - \frac{8}{7x^7} + C$

(b) $\displaystyle\int \frac{x^4+1}{x^6+1}\,dx$ *Hint:* Factor the denominator first, then use the method.

