Walking with a Slower Friend

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Fay and Sam decide to take a walk along their street. Fay's normal walking rate is v_f and Sam's is v_s, with $0 \leq v_s < v_f$. If Fay slows down and walks at Sam's normal rate, then they will be together for the entire trip, but Fay gets little exercise. What if both walk at their normal rates but Fay crosses the street and returns to meet Sam? In order to maximize companionship they must minimize the time, and therefore the distance, between meetings. Here is an optimization problem calculus students will instantly relate to. We present two solutions, one using differentiation, one using only continuity.

The set-up Fay and Sam start together at A (see Figure 1). Sam walks along the left side of the street, while Fay walks to a point B on the right side and then returns to meet Sam at C. Their entire walk then consists of repetitions of this segment.

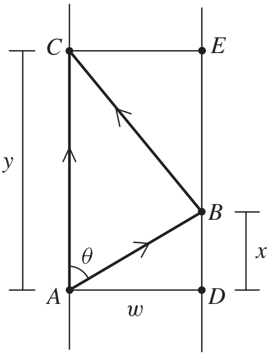

Figure 1. One segment of a walking path.

The time between meetings depends on the choice of x. As an easy example, choose $w = 30$ ft, $v_f = 5$ ft/sec, $v_s = 3$ ft/sec, and $x = 0$ (B and D coincide). Thus Fay walks from A to D and then turns to arrive at C just as Sam arrives at C. Equating their travel times gives the equation

$$\frac{y}{3} = \frac{30 + \sqrt{30^2 + y^2}}{5}.$$

Solving gives $y = 225/4$ ft. For this example $\theta = 90°$ and the time between meetings is $y/v_s = 75/4$ seconds. In contrast, suppose that $x = 40$ ft., so that ADB is a 3-4-5 right triangle. Repeating the earlier analysis reveals that the walkers meet at $y = 195/4$ ft. after $195/12$ seconds, that is 2.5 seconds sooner than in the first example. We seek the value of x that minimizes y in order to minimize the time between meetings.

In general, $AB = \sqrt{w^2 + x^2}$, $BC = \sqrt{w^2 + (y - x)^2}$, and $AC = y$. Equating travel times gives

$$\frac{y}{v_s} = \frac{AB + BC}{v_f} = \frac{\sqrt{w^2 + x^2} + \sqrt{w^2 + (y - x)^2}}{v_f},$$

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and letting $r = v_s/v_f$ (the *velocity ratio*), $0 \le r < 1$, we obtain

$$y = r\left(\sqrt{w^2 + x^2} + \sqrt{w^2 + (y - x)^2}\right). \tag{1}$$

Solution using differentiation Transposing the term $r\sqrt{w^2 + x^2}$ in (1) to the left side, squaring, and collecting terms gives,

$$y^2 - \frac{2r}{1 - r^2}\left(\sqrt{w^2 + x^2} - rx\right) y = 0.$$

Solving for y gives two explicit expressions, $y = 0$ and

$$y = \frac{2r}{1 - r^2}\left(\sqrt{w^2 + x^2} - rx\right). \tag{2}$$

If $y = 0$, then Sam remains at A with $v_s = 0$, while Fay travels back and forth between A and D. We seek the minimal y that satisfies equation (2), with $0 < r < 1$. We first find the critical point x_c, where $dy/dx = 0$, and then show that the corresponding y_c is the global minimum. Differentiating (2) with respect to x gives

$$\frac{dy}{dx} = \frac{2r}{1 - r^2}\left(\frac{x}{\sqrt{w^2 + x^2}} - r\right).$$

Thus the derivative is zero when $r = \frac{x}{\sqrt{w^2+x^2}}$. Squaring both sides and solving for x gives the critical point

$$x = x_c = \frac{rw}{\sqrt{1 - r^2}}.$$

Substituting x_c into (2) gives

$$y_c = \frac{2r}{1 - r^2}\left[\sqrt{w^2 + \frac{r^2 w^2}{1 - r^2}} - \frac{r^2 w}{\sqrt{1 - r^2}}\right] = \frac{2rw}{\sqrt{1 - r^2}}.$$

From the explicit expressions for y and its derivative, we see that y is a continuous function of x and that $\frac{dy}{dx}$ exists for all x. Also $\lim_{x\to\infty} y = \infty$ and $\lim_{x\to-\infty} y = \infty$, since $0 < r < 1$. Hence y_c is the global minimum value of y.

Solution using continuity The optimal solution we found in the previous section is symmetric, that is $AB = BC$. In this section we demonstrate this symmetry without using derivatives. The optimal path is then easily found.

Our problem is reminiscent of the classic milkmaid problem, also called Heron's problem [**1**]. A maid and her cow are equidistant from a straight river. The maid must walk to the river to wash her bucket and then walk to the cow. It can be shown, using geometry, that her path length is minimal if the point on the river is equidistant from the cow and her starting point. We cannot apply this method to our problem for two reasons. First, the meeting point is not prescribed, but is determined as part of the solution. Second, there is a timing issue: there are two walkers and they must meet at the rendezvous point simultaneously. The crux of the argument nevertheless is to show that the solution is symmetric, that is $y = 2x$. In Figure 2, suppose that B corresponds to the optimal value of x, so that the two walkers meet at point C and cannot optimally meet at any point between A and C. With this assumption, the meeting point problem is eliminated, but not the timing issue.

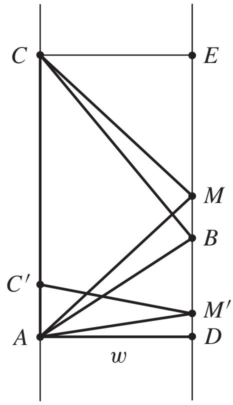

Figure 2.

To proceed, suppose that path ABC is not symmetric ($AB \neq BC$), then path AMC is shorter than the assumed optimal path ABC, if M is the midpoint of DE. Thus if Fay walks the symmetric path AMC she will arrive at C earlier than Sam. We now consider all possible symmetric paths $AM'C'$ that Fay may follow, with M' on DE and C' on AC. At one extreme, for the path ADA with $C' = A$, Fay arrives at the end point later than Sam, who was already there when Fay started out. At the other extreme, for the path AMC with $C' = C$, we have already seen that Fay reaches the end point earlier than Sam. By the intermediate value theorem, there must be a point C' *strictly between* A and C at which Fay and Sam arrive simultaneously. But this contradicts the assumed optimality of the path ABC. We conclude that ABC must be symmetric after all.

Knowing that the path must be symmetric, leads directly to a solution. If y is the shortest distance for Sam, then by symmetry $2\sqrt{w^2 + (y/2)^2}$ is the corresponding distance for Fay. Equating travel times gives

$$\frac{y}{v_s} = \frac{2\sqrt{w^2 + (y/2)^2}}{v_f} \quad \text{or} \quad y^2 = 4r^2(w^2 + y^2/4).$$

So $y = \frac{2rw}{\sqrt{1-r^2}}$, as previously found by calculus.

Optimal initial direction In order to minimize time between meetings, Fay's initial direction will be

$$\theta = \theta_c = \arccos \frac{x_c}{\sqrt{w^2 + x_c^2}} = \arccos(r),$$

where θ is the angle BAC in Figure 1. The time between meetings will be

$$t_c = \frac{y_c}{v_s} = \frac{2rw}{v_s\sqrt{1 - r^2}} = \frac{2w}{\sqrt{v_f^2 - v_s^2}}.$$

The optimal angle θ_c depends on the velocity ratio r and is independent of the street width w. For walkers without calculators, Figure 3 plots θ_c as a function of r.

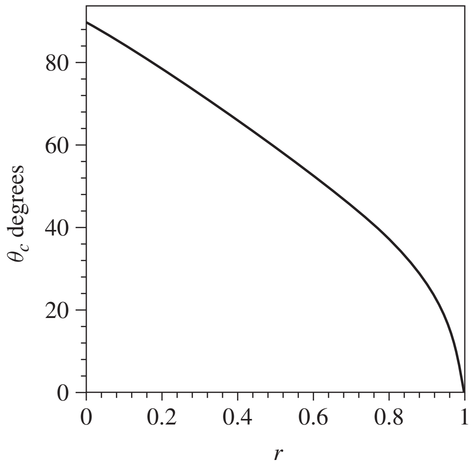

Figure 3. Optimal initial angle versus speed ratio.

For the parameters $w = 30$ ft., $v_f = 5$ ft./sec., $v_s = 3$ ft./sec., as in our example, the optimal solution corresponds to $\theta = \arccos(3/5) \cong 53°$. The distance along AC between meetings will be 45 ft. and the time between meetings will be 15 sec.

Summary. Fay and Sam go for a walk. Sam walks along the left side of the street while Fay, who walks faster, starts with Sam but walks to a point on the right side of the street and then returns to meet Sam to complete one segment of their journey. We determine Fay's optimal path minimizing segment length, and thus maximizing the number of times they meet during the walk. Two solutions are given: one uses derivatives; the other uses only continuity.

Acknowledgment. Our thanks to the reviewers and editors for several very helpful suggestions.

References

1. A. Bogomolny, Heron's Problem; available at http://www.cut-the-knot.org/Curriculum/Geometry/HeronsProblem.shtml

The Cobb-Douglas Function and Hölder's Inequality

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Whenever I teach Business Calculus, I am struck that the Cobb-Douglas production function is ripe for an application of Hölder's inequality [**2**]. This capsule explores an example.

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