CLASSROOM CAPSULES

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Series Involving Iterated Logarithms

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Almost every standard calculus text shows that $\sum \frac{1}{n}$ diverges but that when $p > 1$, $\sum \frac{1}{n^p}$ converges. This is often called the p-test. The p-test always interested us, since we felt that there was not much difference between, say, $\sum \frac{1}{n}$ and $\sum \frac{1}{n^{1.001}}$. Some texts, for example [**4**], go on to interpolate closer divergent/convergent pairs of series: for P greater than 1, the pair $\left(\sum \frac{1}{n \ln n}, \sum \frac{1}{n(\ln n)^P}\right)$ are closer to each other and, for $Q > 1$, the pair $\left(\sum \frac{1}{n \ln n \ln \ln n}, \sum \frac{1}{n \ln n (\ln \ln n)^Q}\right)$ are still closer. By "closer" we mean that $\frac{1}{n} \gg \frac{1}{n \ln n} \gg \frac{1}{n \ln n \ln \ln n} \gg \frac{1}{n \ln n (\ln \ln n)^Q} \gg \frac{1}{n (\ln n)^P} \gg \frac{1}{n^p}$, where $b_n \gg \ell_n$ means that $\lim_{n \to \infty} \frac{b_n}{\ell_n} = \infty$. Furthermore these relations hold even if p is very close to 1 while P (or Q) is very large.

Let $b > 1$ be any base. Denote the iteration of k logarithms, $\log_b \log_b \cdots \log_b x$, by $\log_b^k x$; in particular when $b = e$, we write $\ln \ln \cdots \ln x = \ln^k x$.

Theorem 1. *Let $M = M(b, k)$ be so large that $\log_b^k M > 0$. Then*

$$S = S(b, k, p) = \sum_{n \geq M} \frac{1}{n \log_b n \log_b^2 n \cdots \log_b^{k-1} n (\log_b^k n)^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Notice that the p-test is the case $S(e, 0, p)$, if we interpret $\ln^0 n$ to be n, while the assertion that the other two pairs mentioned above are divergent/convergent pairs are the cases $S(e, 1, p)$ and $S(e, 2, p)$. The well known Integral Test proof of $S(e, 1, p)$ easily extends to a proof of $S(e, k, p)$ for every k. There is a less well known, integration-avoiding, proof for $S(2, k, p)$. We will extend both proofs, thereby giving two proofs of Theorem 1.

As we mentioned above, as k increases, the kth cases of Theorem 1 appear to provide better and better demarcation of a diverge/converge line. This was the basis for our interest in the $b = e$ cases of Theorem 1, when we were learning calculus. This led us to hope that there might be a series that is right on the diverge/converge line.

Unfortunately, there does not exist a smallest divergent series nor a largest convergent one [**1**]. In a way, this says that in a quest to locate a diverge/converge line, we cannot hope to do much better than something like Theorem 1.

Lemma 1. *Let* $r := \log_b c$. *Then for each positive integer* k,

$$\lim_{x\to\infty} \frac{\log_b^k x}{\log_c^k x} = r.$$

Lemma 1 is a not so obvious generalization of the standard logarithmic identity $\frac{\log_b x}{\log_c x} = r$. It will facilitate our proof of Theorem 1.

To prove the lemma, set $L(x) := \log_c x$, so that $\log_b x = \log_b c \log_c x = rL(x)$. Using the convention introduced above for functional iteration, we need to prove

$$\lim_{x\to\infty} \frac{(rL)^k x}{L^k x} = r.$$

But $(rL)^k x = r(Lr)^{k-1}Lx$, so this is equivalent to

$$\lim_{x\to\infty} \frac{(Lr)^{k-1}Lx}{L^k x} = 1.$$

This is true if $k = 1$. Continuing by induction

$$\begin{aligned}(Lr)^k Lx &= L\left[r\left((Lr)^{k-1}Lx\right)\right]\\ &= Lr + L\left\{(Lr)^{k-1}Lx\right\}\\ &= Lr + L\left\{L^k x\,(1+\delta(x))\right\}\\ &= Lr + L^{k+1}x + L\,(1+\delta(x))\\ &= L^{k+1}x + \{Lr + \epsilon(x)\}.\end{aligned}$$

The second and fourth steps are due to L having the multiplicative property. The existence of a function $\delta(x)$ which tends to 0 at ∞ so that the third step holds is a reformulation of the induction hypothesis. Since $L(1) = 0$ and L is continuous at 1, there is a function $\epsilon(x)$ which tends to 0 at ∞ so that the last step holds. Complete the proof by dividing by $L^{k+1}x$ and observing that $\frac{Lr+\epsilon(x)}{L^{k+1}x} \to 0$ as $x \to \infty$ since the numerator is bounded while the denominator becomes infinite.

Proof of Theorem 1. By Lemma 1 and the Limit Comparison Test, it suffices to prove Theorem 1 for any particular choice of base. The Integral Test proof works best for base e. By the Integral Test, it suffices to evaluate

$$\int_M^\infty \frac{dx}{x\ln x\cdots(\ln^k x)^p}.$$

This is done by means of the substitution $u = \ln^k x$.

Our second proof of Theorem 1 avoids the use of the integral calculus. It works best for base 2 and uses the Cauchy Condensation Test. This test says that if $a_n \searrow 0$, then

$\sum a_n$ and $\sum 2^m a_{2^m}$ converge or diverge together; just group the terms of $\sum a_n$ into dyadic blocks to prove it [**2**]. For our second proof: we let

$$a_n = \frac{1}{n \log_2 n \cdots \log_2^{k-1} n (\log_2^k n)^p}.$$

Then $\log_2^j 2^m = \log_2^{j-1}\left(\log_2(2^m)\right) = \log_2^{j-1} m$ and

$$2^m a_{2^m} = 2^m \frac{1}{2^m m \cdots \log_2^{k-2} m (\log_2^{k-1} m)^p},$$

so since the terms of S are eventually decreasing, the k case is equivalent to the $k-1$ case, is equivalent to the $k-2$ case, ..., is equivalent to the $k=0$ case. Thus $S(2,k,p)$ converges if and only if $\sum n^{-p}$ converges. A final application of the Cauchy Condensation Test reduces this to the easy analysis of the geometric series $\sum (2^{1-p})^m$. ■

Some history. The treatment of the natural logarithm cases $S(e,k,p)$ by means of the Integral Test is very well known. For example, they are treated in the classic encyclopedic work of Konrad Knopp, where 4 distinct proofs are given, one of which uses Cauchy's Condensation Test. That particular proof can easily be modified to do the cases $S(b,k,p)$ when $b \geq 2$ but not the cases when $1 < b < 2$ [**3**].

References

1. J. M. Ash, Neither a worst convergent nor a best divergent series exists, this JOURNAL **28** (1997) 296–297.
2. A. L. Cauchy, *Cours d'Analyse de l'École Polytechnique*. Part I Analyse Algébrique, Paris, 1831.
3. K. Knopp, *Theory and Applications of Infinite Series*, trans. of 2nd German ed., Hafner NY, 1971.
4. J. Stewart, *Single Variable Calculus*, 6th ed., Brooks/Cole, 2008, Belmont CA, p. 740, problems 27 and 28.

Sums of Integer Powers via the Stolz-Cesàro Theorem

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One of the powerful rules for evaluating limits of sequences or series is the Stolz-Cesàro theorem which is a discrete form of l'Hôpital's rule.

Stolz-Cesàro Theorem. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. If b_n is positive, strictly increasing and unbounded and the following limit exists:*

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = l,$$

then the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l$$

also exists and it is equal to l.