Sledge-Hammer Integration

Henry F. Ahner

Henry F. Ahner (hahner@adelphi.edu) received his Ph.D. from New York University in 1968. Currently Associate Provost for Administration at Adelphi University, he enjoys the occasional opportunity to teach a college math course.

Ev'ry valley shall be exalted, and ev'ry mountain and hill made low;
the crooked straight, and the rough places plain. Isaiah 40:4

The idea of sledge-hammer integration

Imagine a region in the first quadrant bounded above by a curve $y = f(x)$ and below by the x-axis, and lying between $x = 0$ and $x = 1$. Suppose this region is physically realized by a uniform layer of an incompressible substance, perhaps clay or putty. To calculate the area of the region, pound the curve from above—with a hammer, or some other tool—in an area-preserving way. Local maxima decrease and, because the process is area preserving and constrained by the vertical edges, local minima increase. Protrusions diminish and pot-holes fill up; valleys are exalted and hills made low. Ultimately, the region becomes a rectangle, which, because its base is 1, has height equal to the area under the curve. This is sledge-hammer integration. By re-scaling, it can be applied to any interval. The choice of (0, 1) is dictated by convenience.

This vision of integration, although intuitively obvious, seems not to be in textbooks. It is offered here, together with the analytic formulation that follows, as a supplement to the usual discussion of integration carried out, for example, in introductory calculus classes and texts.

Analytic method

This is integration as averaging. For the first blow of the hammer, specifically average the values of $f(x)$ that are at positions symmetric with respect to $x = 1/2$. In other words, replace $f(x)$ with $g(x) = (f(x) + f(1-x))/2$. An example is given in Figure 1, where the first plot (top-left) gives the curve $f(x) = x^2$. In the second plot (top-right), both $f(x)$ and $f(1-x)$ are shown and in the third plot (middle-left) g is added. These three curves all enclose the same area. The newest curve in each plot is drawn heavier, for easy identification.

By construction, g is symmetric with respect to $x = 1/2$. Further flatten $f(x)$ by taking the first half of g (from 0 to 1/2) and stretch it to the full interval (0, 1) by substituting $x/2$ for x. This is done in the fourth plot of Figure 1.

To summarize, we propose to average a given function $f_0(x) = f(x)$ by replacing it with

$$f_1(x) = \frac{f_0(x/2) + f_0(1 - x/2)}{2},$$

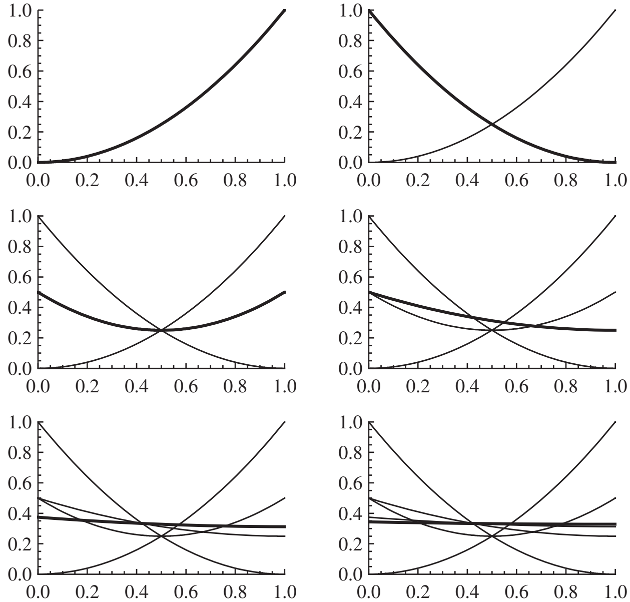

Figure 1.

which is the first step of an iterative process defined by

$$f_n(x) = \frac{f_{n-1}(x/2) + f_{n-1}(1 - x/2)}{2}.$$

Successive functions will be increasingly flatter versions of the initial function.

Two further steps of our example, $f_2(x)$ and $f_3(x)$, are added in the bottom two plots of Figure 1. All six curves in the final plot have the same integral. Note that f_3 is nearly a constant, namely $1/3$, which is the value of the integral of f from 0 to 1.

A second example is presented in Figure 2 with $f(x) = \text{Abs}(x + \text{Cos}(17x))$. This plot shows several iterations of our averaging process applied to f (depicted by the solid curve). Iterations f_1 to f_4 are shown by dashed lines, with dash-width progressively longer for higher iteration indices. As a third example, if $f_0 = \text{Cos}(ax + b)$,

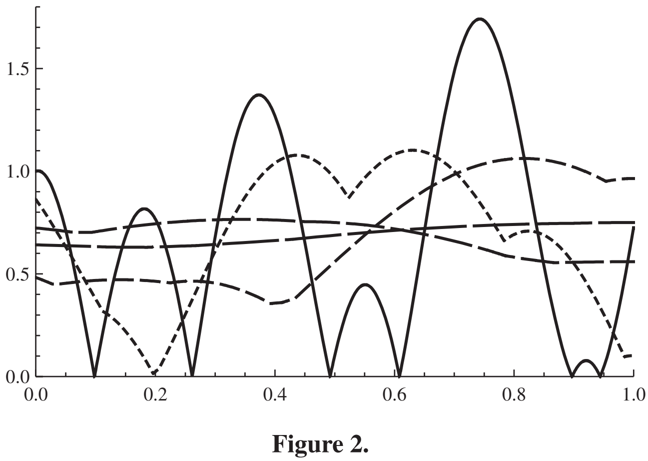

Figure 2.

then $f_1 = \text{Cos}((a+2b)/2)\,\text{Cos}(a(1-x)/2)$; note wavelength doubling means fewer valleys.

At this stage, we could prove a theorem regarding the convergence of the algorithm to the desired integral, but there is a better way to convince students that this method successfully approximates the integral of f from 0 to 1.

Why it works

Consider a specific stage of the process, for example, the third iteration. In terms of $f(x)$, $f_3(x)$ has the form,

$$f_3(x) = \frac{1}{8}\left(f\left(\frac{x}{8}\right) + f\left(\frac{1}{4}-\frac{x}{8}\right) + f\left(\frac{1}{4}+\frac{x}{8}\right) + f\left(\frac{1}{2}-\frac{x}{8}\right) + f\left(\frac{1}{2}+\frac{x}{8}\right) + f\left(\frac{3}{4}-\frac{x}{8}\right) + f\left(\frac{3}{4}+\frac{x}{8}\right) + f\left(1-\frac{x}{8}\right)\right).$$

The values at which each f is evaluated vary as x ranges from 0 to 1. These values are depicted in Figure 3, which shows that although f_3 usually averages eight values of f, the endpoints are exceptions.

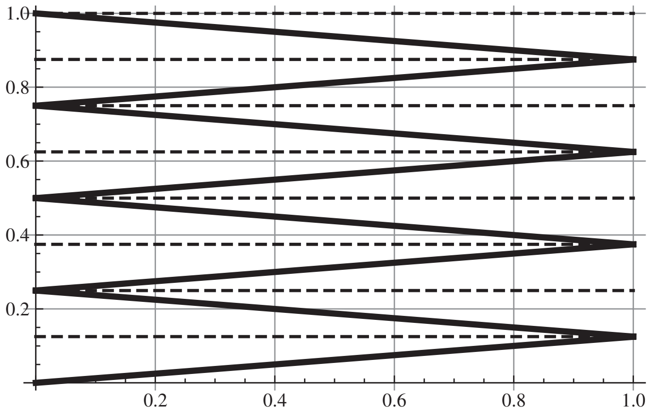

Figure 3.

Specifically, when $x = 0$ we get,

$$f_3(0) = \frac{1}{8}\left(f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right),$$

which is the *trapezoid* method applied to f (with four subdivisions). In contrast, when $x = 1/2$, we get,

$$f_3\left(\frac{1}{2}\right) = \frac{1}{8}\left(f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + f\left(\frac{7}{16}\right) + f\left(\frac{9}{16}\right) + f\left(\frac{11}{16}\right) + f\left(\frac{13}{16}\right) + f\left(\frac{15}{16}\right)\right).$$

This is the *mid-point* method applied to f (with eight subdivisions). Finally, at $x = 1$,

$$f_3(1) = \frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right),$$

which is the mid-point method (with four subdivisions). Of course, both trapezoid and mid-point methods are well-known numerical integration techniques.

This is perfectly general. For all positive integers n, $f_n(0)$ gives a trapezoid method approximation; while $f_n(1/2)$ and $f_n(1)$ express the mid-point method. Other evaluations of f_n are particular Riemann sums. This shows clearly why sledge-hammer integration works.

Acknowledgment. Graphs prepared using *Mathematica*. Gratefully acknowledged are discussions with M. B. Ahner, and editor's and reviewer's constructive suggestions.

Brown Sharpie