An Infinite Series for π with Determinants

NOBUO ADACHI
Millersville University
Millersville, PA 17551

There are a variety of expressions for π. I have discovered an unusual one using determinants.

THEOREM.

$$\frac{\pi}{2}-1=2^0\begin{vmatrix} 1 & 0 & 1 \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 1 \\ 0 & 1 & 1 \end{vmatrix}+2^1\begin{vmatrix} 1 & 0 & 1 \\ \frac{1}{2}\sqrt{2+\sqrt{2}} & \frac{1}{2}\sqrt{2-\sqrt{2}} & 1 \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 1 \end{vmatrix}$$

$$+2^2\begin{vmatrix} 1 & 0 & 1 \\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}} & \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}} & 1 \\ \frac{1}{2}\sqrt{2+\sqrt{2}} & \frac{1}{2}\sqrt{2-\sqrt{2}} & 1 \end{vmatrix}+\cdots$$

$$+2^{n-1}\begin{vmatrix} 1 & 0 & 1 \\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}_n}}} & \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}_n}}} & 1 \\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}_{n-1}}}} & \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}_{n-1}}}} & 1 \end{vmatrix}$$

$$+\cdots.$$

This can be proved with some geometry and calculus. A quarter of the area of a circle of unit radius can be expressed as the sum of an infinite series:

$$\frac{\pi}{4}=\Delta_0+2^0\Delta_1+2^1\Delta_2+2^2\Delta_3+\cdots+2^{n-1}\Delta_n+\cdots,$$

where the Δ_i $(i=0,1,2,\dots)$ are the areas of the isosceles triangles indicated in FIGURE 1. Let Q and P_0 be the points $(1,0)$ and $(0,1)$, respectively, and P_n be the apex of the triangle of area Δ_n $(n\geqslant 1)$.

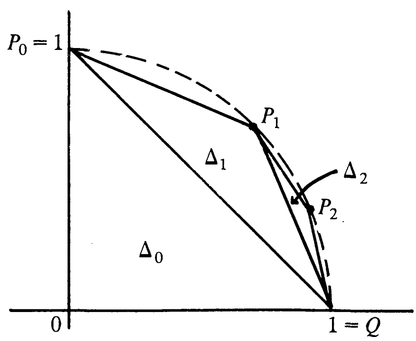

FIGURE 1

Recall that the area of the triangle with vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ listed in counterclockwise order is given by the determinant

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Since the nth triangle has vertices Q, P_n, and P_{n-1}, it is sufficient to identify the coordinates of each P_n. For each $n \geqslant 1$, P_n is the midpoint of the circular arc from P_{n-1} to Q. Hence, the tangent to the circle through P_n is parallel to the secant $P_{n-1}Q$. Rewriting this fact in analytic terms will provide the information we seek.

Let the coordinates of P_n be (u_n, v_n). The equation of the quarter circle is $y = f(x)$ $(0 \leqslant x \leqslant 1)$, where $f(x) = \sqrt{1-x^2}$. Since

$$f'(u_n) = \frac{f(u_{n-1}) - f(1)}{u_{n-1} - 1} \qquad (n \geqslant 1),$$

we find that

$$\frac{u_n}{\sqrt{1-u_n^2}} = \frac{\sqrt{1-u_{n-1}^2}}{u_{n-1}-1}.$$

Squaring both sides and solving for u_n yields $u_n^2 = \frac{1}{2}(1 + u_{n-1})$, which, with $u_n^2 + v_n^2 = 1$, produces $v_n^2 = \frac{1}{2}(1 - u_{n-1})$. Since u_n and v_n are both nonnegative and $u_0 = 0$, we obtain in succession

$$(u_0, v_0) = (0,1)$$

$$(u_1, v_1) = \left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$$

$$(u_2, v_2) = \left(\sqrt{\frac{1}{2}\left(1+\frac{\sqrt{2}}{2}\right)}, \sqrt{\frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right)}\right) = \left(\frac{1}{2}\sqrt{2+\sqrt{2}}, \frac{1}{2}\sqrt{2-\sqrt{2}}\right)$$

$$(u_3, v_3) = \left(\sqrt{\frac{1}{2}\left(1+\frac{1}{2}\sqrt{2+\sqrt{2}}\right)}, \sqrt{\frac{1}{2}\left(1-\frac{1}{2}\sqrt{2+\sqrt{2}}\right)}\right)$$

$$= \left(\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}, \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}\right)$$

$$\vdots$$

$$(u_n, v_n) = \left(\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}, \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}\right).$$

Thus the area of the nth triangle is

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ u_n & v_n & 1 \\ u_{n-1} & v_{n-1} & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0. & 1 \\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{\cdots\sqrt{2}}}}_n & \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{\cdots\sqrt{2}}}}_n & 1 \\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{\cdots\sqrt{2}}}}_{n-1} & \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{\cdots\sqrt{2}}}}_{n-1} & 1 \end{vmatrix},$$

from which the desired result is obtained.

I wish to thank Associate Editor Edward J. Barbeau for his helpful suggestions in revising the manuscript. This paper is dedicated to Mary K. Geffert.

References

[1] Petr Beckman, A History of π, St. Martin's Press, New York, 1971.
[2] Y. Mikami, The Development of Mathematics in China and Japan, Chelsea, New York, 1961, pp. 200–203, 213–217.