NOTES

Heads Up: No Teamwork Required

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Many team games require collaborative strategies. In this Note, I present a game that paradoxically requires cooperation, but at the same time forbids communication of any kind. A team skilled in probability can compete as if they could confer (almost).

In the "Hat Problem" [**1**], each contestant on a team tries to guess the color (blue or red) of a hat on his/her head while seeing only the hats of the other contestants. The guesses are made simultaneously, and "passing" is an option. The team wins if at least one person guesses and no one guesses incorrectly. Before the guessing round, the participants are allowed to discuss strategy. With best play, an n-person team (where n is one less than a power of 2) can win with the surprisingly high probability $n/(n+1)$. (The solution to the Hat Problem provides a novel perspective on Hamming codes, but you don't need to know anything about Hamming codes to understand this note.) I enjoyed the Hat Problem and was investigating variations of it when I thought of eliminating both the pregame strategy meeting and any information the teammates could gain from each other. The resulting game, recast in the context of coin-flipping, has some curious mathematical features.

> **Heads Up:** A team of $n \geq 2$ people is selected to participate in a game in which the team may win a prize. The players are not allowed to communicate with each other before or during the game. (Assume, for reasons that should become clear in a moment, that the players have never met before and know nothing about each other.) Each player is given a fair coin. At a signal, the players simultaneously open their hands to reveal either the coin or nothing. Those players showing coins then flip their coins. The team wins if at least one coin is flipped and all the flipped coins land heads. What strategy should each player follow in order to maximize the likelihood of the team winning, and what is the probability of winning?

The crucial factor in Heads Up is that the team members cannot communicate with each other so they cannot form a strategy together. (In the solution to the Hat Problem, the players determine an ordering of themselves so that they can convert the hat color distribution into a vector.) Ideally, the team would choose one member to flip the coin (and the team would win half of the time), but this is impossible because of the ban on communication. Are there, then, only two alternative strategies for each player: either flip the coin or don't flip it? If so, then not flipping the coin isn't good because if everyone does that, the team loses. But if everyone flips the coin, the team wins with very low probability ($(1/2)^n$). With $n = 2$, the probability of winning if both players flip the coin is $1/4$. However, with a superior strategy a two-person team can do better than $1/4$. In fact, the same is true for any number of players. With best play, an n-person team can do better than they could if they were able to choose two members to flip their coins!

Since the problem is about probabilities, we suppose that the players approach their strategies probabilistically. In addition, the players must assume that their teammates will also arrive at this conclusion. Each player should flip the coin with probability p, where the value of p is to be determined ($0 \le p \le 1$). We obtain the probability of the team winning, $w_n(p)$, via a binomial expansion:

$$\begin{aligned} w_n(p) &= \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \binom{n}{k} p^k (1-p)^{n-k} \\ &= \left(\frac{p}{2} + 1 - p\right)^n - (1-p)^n = \left(1 - \frac{p}{2}\right)^n - (1-p)^n. \end{aligned} \tag{1}$$

Using calculus, we find that w_n is maximized at

$$p_n \equiv p = \frac{2^{n/(n-1)} - 2}{2^{n/(n-1)} - 1}, \tag{2}$$

and the maximum winning probability is

$$w_n(p_n) = \left(2^{n/(n-1)} - 1\right)^{1-n}. \tag{3}$$

For example, with two players, $p_2 = 2/3$ and $w_2(p_2) = 1/3$. The coin can be used to generate an event with the required probability: express p_n in base 2 and flip the coin until the number generated (heads = 1, tails = 0) deviates from p_n. (This concept was the subject of a Putnam Competition problem [**2**, p. 137].) The case $n = 2$ can be handled in a particularly simple way: flip the coin twice; if it lands tails both times, do over; the probability of one heads and one tails is then $2/3$.

Let's analyze our solution to Heads Up. Although we might conjecture that the maximum probability of winning in (3) tends to 0 as n tends to infinity, a calculation with l'Hôpital's rule reveals that

$$\lim_{n\to\infty} w_n(p_n) = \frac{1}{4}. \tag{4}$$

Actually, we might have guessed this result! Observing that in the formula in (1) the second quantity is almost the square of the first quantity, we could reason that the limit has the form $x - x^2$, and everyone who can complete a square knows that the maximum value of $x - x^2$ is $1/4$.

It seems reasonable that the maximum probability of winning in (3) is a decreasing function of n. Numerical calculations indicate that this is true, and our intuition about the problem reinforces it (the chances of the team winning would appear to decrease as n increases). Let us define a real-variable version of the function:

$$f(x) = \left(2^{x/(x-1)} - 1\right)^{1-x}, \quad x > 1.$$

It is a tricky problem to prove that $f(x)$ is decreasing. Here is a calculus proof found by my undergraduate student Khang Tran. Take a logarithm:

$$g(x) = \ln f(x) = (1-x)\ln\left(2^{x/(x-1)} - 1\right).$$

Now

$$g'(x) = -\ln\left(2^{x/(x-1)} - 1\right) + \frac{2^{x/(x-1)}\ln 2}{(x-1)(2^{x/(x-1)} - 1)},$$

and we want to show that $g'(x)$ is negative for $x > 1$. Make a change of variables, $y = 1/(x-1)$:

$$h(y) = -\ln\left(2^{y+1}-1\right) + \frac{2^{y+1}y\ln 2}{2^{y+1}-1} = -\ln\left(2^{y+1}-1\right) + y\ln 2 + \frac{y\ln 2}{2^{y+1}-1}.$$

It suffices to show that $h(y)$ is negative for $y > 0$. Observe that $h(0) = 0$ and

$$h'(y) = -\frac{2^{y+1}\ln 2}{2^{y+1}-1} + \ln 2 + \frac{\ln 2}{2^{y+1}-1} + \frac{-2^{y+1}y(\ln 2)^2}{(2^{y+1}-1)^2}.$$

The first three terms sum to zero and the last term is negative.

Surprisingly, we can give a short proof that $w_n(p_n)$ is decreasing via the solution to Heads Up. Here it is:

$$w_n(p_n) \geq w_n(p_{n+1}) = w_{n+1}(p_{n+1}). \tag{5}$$

The inequality in (5) holds by virtue of the definition of p_n. The equality, which can be proved directly by combining (1) and (2), is an algebraic curiosity; it says that, for each n, the maximum value of w_{n+1} occurs on the graph of w_n. Put another way, this peculiar identity says that the success probabilities for n and $n+1$ are equal for the optimum probability p_{n+1}. This is like one team member "dropping out" (but, of course, no one member can drop out).

We see from (2) that the optimum probability p_n decreases to 0 as n tends to infinity. What can we say about $n\,p_n$, the expected number of coins flipped? From (2), using l'Hôpital's rule, we determine the limiting value:

$$\lim_{n\to\infty} n\,p_n = \ln 4. \tag{6}$$

(We could have guessed this result by invoking the fact that $\lim_{n\to\infty}(1-c/2n)^n = e^{-c/2}$ and recognizing that $x - x^2$ is maximized when $x = 1/2$.)

Furthermore, the expected number of coins flipped increases with n; that is,

$$n\,p_n < (n+1)p_{n+1}, \quad n \geq 2, \tag{7}$$

as we shall prove. Does intuition support this result? A calculus proof, with n replaced by a real variable x, is possible but tedious (make the change of variables $y = 2^{x/(x-1)}$ and differentiate). We prefer to stay with the discrete variable. The inequality (7) is equivalent to

$$n \cdot \frac{2^{n/(n-1)}-2}{2^{n/(n-1)}-1} < (n+1)\cdot\frac{2^{(n+1)/n}-2}{2^{(n+1)/n}-1},$$

which is equivalent to

$$\frac{1}{n+1}\cdot\frac{2^{(n+1)/n}-1}{2^{1/n}-1} < \frac{1}{n}\cdot\frac{2^{n/(n-1)}-1}{2^{1/(n-1)}-1},$$

and, by the formula for the sum of a geometric series, to

$$\frac{1+2^{1/n}+2^{2/n}+\cdots+2^{n/n}}{n+1} < \frac{1+2^{1/(n-1)}+2^{2/(n-1)}+\cdots+2^{(n-1)/(n-1)}}{n}. \tag{8}$$

This last inequality has a pleasing form. It's a consequence of a simple proposition on convex functions (applied to the function $f(x) = 2^x$).

PROPOSITION. *Let f be a convex function defined on the interval* $[0, 1]$. *Then, for* $n \geq 2$,

$$\frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{i}{n}\right) \leq \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n-1}\right).$$

If f is strictly convex, then this inequality is strict.

This proposition is easy to prove using Jensen's inequality for convex functions: $f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b)$, for $0 \leq \lambda \leq 1$.

Returning to the Hat Problem, we ask the question, what happens if there is no pregame strategy meeting (and this is the only change we make)? In the case $n = 3$, no prearranged ordering of players is needed, and the team wins with probability $3/4$. (A player seeing two hats of the same color guesses the other color, and otherwise passes.) For $n > 3$, is the winning probability merely $w_n(p_n)$, or is there a way for the players to use the knowledge of each other's hat colors?

Acknowledgment. I am grateful to Suren Fernando, Upendra Kulkarni, Anthony Vazzana, and the referees for their suggestions concerning this note.

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The Humble Sum of Remainders Function

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The sum of divisors function is one of the fundamental functions in elementary number theory. In this note, we shine a little light on one of its lesser-known relatives, the sum of remainders function. We do this by illustrating how straightforward variations of the sum of remainders can 1) provide an alternative characterization for perfect numbers, and 2) help provide a formula for sums of powers of the first n positive integers. Finally, we give a brief discussion of perhaps why the sum of remainders function, despite its usefulness, is less well known than the sum of divisors function.

Some notation is in order. The standard notation [**3**] for the sum of divisors function is $\sigma(n)$:

$$\sigma(n) = \sum_{d|n} d.$$

We denote the sum of remainders function by $\rho(n)$, namely,

$$\rho(n) = \sum_{d=1}^{n} (n \bmod d).$$