Chaos and Love Affairs

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In [**1**] and [**2**], Steven Strogatz explores the possible dynamics of a two-dimensional system of linear differential equations by interpreting the variables R and J as Romeo's and Juliet's love for each other, respectively. The dynamics is governed by the equations $\dot{R} = aR + bJ$ and $\dot{J} = cR + dJ$, where the constant coefficients a, b, c, d are real numbers, which may be positive or negative. Each combination of signs correspond to a possible human behavior; for example, a "cautious lover" has $a < 0$, $b > 0$. Different values of these coefficients give rise to different dynamics, which can be studied by means of fixed points and linear stability analysis. For these, one can find amusing (or tragic) interpretations, such as love-hate cycles, mutual indifference (fixed point at the origin), and love feasts or war (trajectories that go off to infinity).

In this note I examine the following question: How complicated does a model of love affairs need to be in order to exhibit chaos? To do this, I will summarize the necessary conditions for continuous time models with increasing number of variables. A 1-variable model needs time-delay (external periodic driving is not enough); a 2-variable model needs external periodic driving; and a 3-variable model just needs one nonlinear term (no external forcing is needed). The R–J space interpretations of these conditions suggest a rather simple mnemonic device for students to remember them. In more detail, the necessary conditions for chaos in continuous time systems are as follows:

1. A one-dimensional nonlinear time-delayed equation of the form $\dot{x} = f(x - \tau)$ can be chaotic. This requires defining $f(t)$ in the interval $-\tau < t < 0$, which in principle makes the system infinite-dimensional. In practice one observes chaotic attractors with an effective dimensionality that increases with τ; see the discussion in [**3**, Ch. 8], and references therein. The best known example of this situation is the Mackey–Glass equation for the regeneration of blood cells in patients with leukemia.
2. For a two-dimensional autonomous system, the Poincaré–Bendixson theorem implies that only closed orbits can occur. These are limited to fixed points, limit cycles, and cycle graphs. Therefore, there can be no chaos: see [**2**, pp. 203–210]. However, a system of two nonlinear differential equations with external forcing can exhibit chaos. There is a nice study of the forced nonlinear pendulum in [**4**, Ch. 3], which includes several plots of strange attractors. A forced linear pendulum can exhibit resonance, but not chaos.
3. A system of three or more coupled differential equations with just one nonlinear term (for example, a product of two variables), can have bundles of trajectories that flow, stretch, and fold at the same time; this can result in bounded, deterministic nonperiodic flow with sensitive dependence to initial conditions, in a word, chaos. The simplest example, called the Rössler attractor, is discussed in [**2**, Ch. 12], along with more complicated and better-known examples such as the Lorenz system.

While in no way do I want to imply a rigorous connection between the above statements and the R–J psychological space or its equivalent in other dimensions, they suggest the following mnemonic device to remember when chaos can occur:

To have chaos you need a single person obsessed with the past, a couple affected by the phases of the moon, or a ménage à trois.

It would be a nice exercise to incorporate into this sentence the conditions necessary for chaos in discrete-time dissipative systems, and for Hamiltonian chaos in general; however, I have been unable to find proper interpretations for these in R–J space.

REFERENCES

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Edge-Length of Tetrahedra

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For a tetrahedron Δ in Euclidean 3-space, with vertices P, Q, R, S, we define the edge-length $E(\Delta)$ as the sum of the lengths of the six edges $PQ, PR, \ldots$. Let two tetrahedra Δ and Δ' with vertices A, B, C, D and A', B', C', D' be given, with Δ' contained in Δ. We shall show that the relation

$$E(\Delta') < \tfrac{4}{3} \cdot E(\Delta) \tag{1}$$

holds, and that the constant $4/3$ is sharp in the sense that there are pairs Δ, Δ' with $E(\Delta')$ as close to $4/3 \cdot E(\Delta)$ as we want. In particular $E(\Delta')$ can be larger than $E(\Delta)$!

We begin by considering degenerate tetrahedra δ with vertices a, b, c, d (not necessarily different from each other) lying in that order on some straight line in our space. The concept edge-length applies to these in the obvious way. The segment ab is covered three times, by ab, ac, and ad; similarly for cd. But bc is covered four times, by ad, ac, bd, and bc. Thus (with $|ab|$ denoting the length of ab) we have the inequality

$$3|ad| \le E(\delta) \le 4|ad|. \tag{2}$$

The extremes can occur. For equality on the left we must have $b = c$; for that on the right we must have $a = b$ and $c = d$.

Next take any segment AB in space, and consider its orthogonal projection $(AB)_l$ on any directed line l through the origin. The set of these lines has a natural