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How to Determine Your Gas Mileage

ALEXANDER J. IZZO
Bowling Green State University
Bowling Green, OH 43403

Introduction Suppose we want to determine the gas mileage of a car. By definition,

$$\text{gas mileage} = \frac{\text{miles driven}}{\text{gallons of gas used}}.$$

So all we have to do is fill up the tank, drive around a bit, refill the tank, and then divide the miles driven by the amount of gas we put into the tank the second time. The trouble with this approach is that gas mileage varies a great deal with driving conditions. In particular, gas mileage in the city is generally much lower than that on the highway. We really want to determine two gas mileages: one for the city and one for the highway. If each time between filling the tank, the car is driven only in the city or only on the highway, then there isn't any problem, but generally between fuelings we do both city and highway driving. We want a method that can handle this situation. We will present such a method below using least squares. This method is suitable for inclusion in a linear algebra course and is quite satisfactory in practice. However, it is not optimal. Improving the method will lead us naturally to the introduction of weighted least squares. Good references for the elementary linear algebra used in this paper include **[1]** and **[2]**.

To obtain the data needed to apply the method described below, the fuel tank must sometimes be filled completely so that we know how much gas has been used. In addition, the driver must record:

(i) the amount of gas put into the gas tank at each fueling;
(ii) the odometer reading each time the tank is filled;
(iii) the odometer reading before and after each highway trip.

From these records we obtain the required data, namely, the amount of gas used between the $(j-1)^{th}$ and j^{th} times the tank is filled and the distances driven in the city and on the highway during that interval. We will use the following notations:

g_j = gallons of gas used between the $(j-1)^{th}$ and j^{th} fillings of the tank;
c_j = miles driven in the city during that interval;
h_j = miles driven on the highway during that interval;
y_1 = gas mileage in the city;
y_2 = gas mileage on the highway.

Our goal is to determine the values of the unknowns y_1 and y_2 from the data g_j, c_j, and h_j.

The method By the definition of gas mileage, the amount of gas used between the $(j-1)^{th}$ and j^{th} times the tank is filled should be $(c_j/y_1)+(h_j/y_2)$. Thus we obtain one equation for each time the tank is refilled. If the tank is refilled m times, we obtain the following system of m equations in the unknowns y_1 and y_2

$$\frac{c_1}{y_1}+\frac{h_1}{y_2}=g_1, \quad \ldots, \quad \frac{c_m}{y_1}+\frac{h_m}{y_2}=g_m.$$

Letting $x_1 = 1/y_1$ and $x_2 = 1/y_2$, we obtain the linear system

$$\begin{aligned} c_1x_1 + h_1x_2 &= g_1 \\ &\vdots \\ c_mx_1 + h_mx_2 &= g_m. \end{aligned} \tag{1}$$

Setting

$$A=\begin{pmatrix} c_1 & h_1 \\ \vdots & \vdots \\ c_m & h_m \end{pmatrix}, \quad x=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{and} \quad g=\begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix},$$

we can rewrite the system in matrix form as $Ax=g$.

As every good linear algebra student knows, the vectors that can be written in the form Ax are precisely those in the column space of A, so our system has a solution if and only if g is in the column space of A. But the column space of A is at most two-dimensional, while g lies in m-space, so the vector g is almost certain to lie outside the column space of A, as shown in FIGURE 1 below.

Given a system of equations with no solution, it is natural to look for a value of x that makes the difference between the right and left sides as small as possible. That is, we look for a vector $\bar{x}$ that minimizes the (euclidean) length of $A\bar{x}-g$. This is the method of *least squares*. As shown in FIGURE 1, the length of $A\bar{x}-g$ is minimized when $A\bar{x}-g$ is perpendicular to the column space of A.

It is easy to see that a vector is perpendicular to the column space of A if and only if it lies in the null space of A^t. Therefore, we have

$$A^t(A\bar{x}-g)=0 \quad \text{or equivalently} \quad A^tA\bar{x}=A^tg. \tag{2}$$

It is not difficult to see that the matrix A^tA has the same null space as A. So if the columns of A are linearly independent, then the null spaces of A and A^tA each contain only the zero vector, and hence the square matrix A^tA is invertible. Thus,

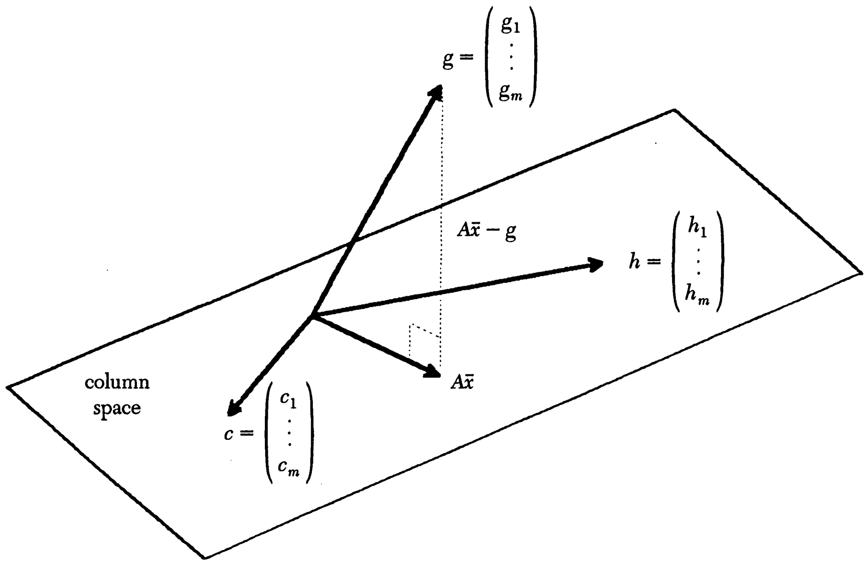

FIGURE 1

assuming that the columns of A are linearly independent, we arrive at our solution:

$$\bar{x} = (A^tA)^{-1}A^tg.$$

If we set $c = (c_1, \ldots, c_m)$, $h = (h_1, \ldots, h_m)$, and $g = (g_1, \ldots, g_m)$, then we can write the solution in terms of dot products as

$$\bar{x} = \begin{pmatrix} c\cdot c & c\cdot h \\ c\cdot h & h\cdot h \end{pmatrix}^{-1} \begin{pmatrix} c\cdot g \\ h\cdot g \end{pmatrix} = \begin{pmatrix} \dfrac{(h\cdot h)(c\cdot g) - (c\cdot h)(h\cdot g)}{(c\cdot c)(h\cdot h) - (c\cdot h)^2} \\ \dfrac{(c\cdot c)(h\cdot g) - (c\cdot h)(c\cdot g)}{(c\cdot c)(h\cdot h) - (c\cdot h)^2} \end{pmatrix}.$$

Recalling that the gas mileages y_1 and y_2 were the reciprocals of x_1 and x_2 and writing out the dot products as sums, we obtain completely explicit formulas for the gas mileages:

$$\begin{aligned} y_1 &= \frac{1}{x_1} = \frac{\Sigma c_j^2 \Sigma h_j^2 - (\Sigma c_j h_j)^2}{\Sigma h_j^2 \Sigma c_j g_j - \Sigma c_j h_j \Sigma h_j g_j} \\ y_2 &= \frac{1}{x_2} = \frac{\Sigma c_j^2 \Sigma h_j^2 - (\Sigma c_j h_j)^2}{\Sigma c_j^2 \Sigma h_j g_j - \Sigma c_j h_j \Sigma c_j g_j}. \end{aligned} \tag{3}$$

We assumed above that the columns of A were linearly independent. If the columns of A are linearly *dependent*, then $\bar{x}$ is not uniquely determined by equation (2), and in fact the set of solutions to equation (2) forms a line, so it is not possible to obtain useful values for the city and highway gas mileages. Of course if m is much bigger than 1, then the columns of A are very unlikely to be linearly dependent.

I presented the problem of determining gas mileage in a linear algebra class at the University of Michigan, and it was very successful. I covered the method of least

TABLE 1 GAS MILEAGE DATA

j	c_j	h_j	g_j	j	c_j	h_j	g_j	j	c_j	h_j	g_j
1	15	283	7.017	18	134	224	10.592	35	207	83	9.074
2	116	171	8.051	19	199	182	10.246	36	52	312	9.519
3	2	367	10.062	20	284	0	10.007	37	76	558	14.841
4	2	300	7.430	21	178	0	6.458	38	0	132	3.148
5	221	69	9.645	22	153	146	9.476	39	11	539	12.356
6	93	215	9.907	23	49	180	6.161	40	194	138	9.832
7	130	87	8.599	24	338	0	13.042	41	334	0	9.872
8	1	360	10.100	25	148	87	7.471	42	304	45	10.786
9	49	219	8.65	26	28	282	9.004	43	147	225	11.234
10	0	273	7.700	27	144	0	5.855	44	0	359	10.200
11	41	268	9.810	28	0	298	6.929	45	129	97	7.824
12	17	321	9.310	29	35	528	14.912	46	193	132	12.032
13	76	613	17.120	30	1	363	8.490	47	256	0	10.762
14	193	0	9.088	31	26	173	5.954	48	194	0	8.507
15	79	178	9.208	32	290	0	14.341	49	41	327	10.310
16	228	0	10.107	33	125	412	15.922	50	233	84	11.113
17	157	191	10.711	34	129	138	8.577				

TABLE 2 GAS MILEAGE RESULTS

Method	City (mi/gal)	Highway (mi/gal)
Unweighted method	26.13	40.02
Weighted method ($a = b = 1$)	25.75	39.47
"Optimal" weighted method ($a = 39.41$, $b = 25.79$)	25.79	39.41

squares for solving a system of linear equations in general and indicated how it was relevant to determining gas mileage but left it to the students to apply the method to obtain explicit formulas. Then, for homework, I gave the students data that I had collected and had them actually determine my gas mileage. The data set used, together with calculated results, are given in Tables 1 and 2 above. (The weighted method is explained below.) The problem of determining gas mileage is an excellent application to cover in a linear algebra class because it is of genuine interest (I really wanted to know the answer!) and because it reinforces many important ideas in linear algebra, including the column space, the null space, linear independence, orthogonality, and the question of when a linear system has a solution.

A better method The method just discussed is quite satisfactory in practice, but it is not optimal. A similar but better method, described below, makes use of inner products other than the standard dot product. While the improved method cannot be included in a course in which the only inner product considered is the dot product, the method would be especially suitable for a course that treats general inner products, since it shows that they arise naturally and are genuinely useful.

To see why the method considered so far is not completely satisfactory consider the special case in which each time between filling the tank the car is driven only in the city or only on the highway. In this case we know how much gas was used in the city and how much on the highway. Hence we can calculate the average number of miles driven per gallon of gas for each type of driving, and clearly these averages are the "correct" values for the gas mileages. That is, we should have

$$y_1 = \frac{\sum c_j}{\sum\limits_{c_j \neq 0} g_j} \quad \text{and} \quad y_2 = \frac{\sum h_j}{\sum\limits_{h_j \neq 0} g_j}. \tag{4}$$

However, since for each j either $c_j = 0$ or $h_j = 0$, our solution (3) gives

$$y_1 = \frac{\sum c_j^2}{\sum c_j g_j} \quad \text{and} \quad y_2 = \frac{\sum h_j^2}{\sum h_j g_j}.$$

Thus, in this special case, the method of least squares gives a *weighted* average with an *inappropriate weighting*.

We rectify this by using *weighted* least squares. Recall that we want to "solve" the linear system (1). Let us call the left hand side of the j^{th} equation the amount of gas predicted to have been used during the j^{th} interval. (Of course the amount of gas *actually* used during the j^{th} interval is g_j.) It is natural to require that the total amount of gas predicted to have been used equal the total amount of gas actually used. (If we were computing just *one* gas mileage we would take the average which is determined by exactly this kind of requirement.) That is, we require that $\Sigma(c_j x_1 + h_j x_2) = \Sigma g_j$. Setting $r_j = c_j x_1 + h_j x_2 - g_j$, this becomes $\Sigma r_j = 0$. If we let $r = (r_1, \ldots, r_m)$, then, in fact, $r = Ax - g$. The method of least squares described above minimizes Σr_j^2 by requiring that r be orthogonal to the columns of A:

$$r \cdot c = 0 \quad \text{and} \quad r \cdot h = 0.$$

The condition $\Sigma r_j = 0$ that we now wish to impose says that r is orthogonal to the vector $(1, \ldots, 1)$. So now we will choose a new inner product $\langle , \rangle$ on $\mathbb{R}^m$ with the property that

$$\text{if } \langle r, c \rangle = 0 \text{ and } \langle r, h \rangle = 0, \quad \text{then } r \cdot (1, \ldots, 1) = 0. \tag{5}$$

Recall that every positive definite symmetric $m \times m$ matrix Q induces an inner product on $\mathbb{R}^m$ by the equation $\langle v, w \rangle = v^t Q w$, and that every inner product arises in this way. Clearly, we should choose an inner product whose corresponding matrix Q is as simple as possible, and this suggests trying to make Q diagonal. The reader can easily check that the inner product induced by a diagonal matrix Q will satisfy (5) provided Q has the form

$$Q = \begin{pmatrix} \dfrac{1}{ac_1 + bh_1} & & 0 \\ & \ddots & \\ 0 & & \dfrac{1}{ac_m + bh_m} \end{pmatrix}$$

where a and b are arbitrary positive numbers. (We require a and b to be positive in order to insure that Q is positive definite.)

Now we carry out essentially the same procedure as before, except that we use the inner product induced by Q instead of the dot product. Our solution will be the vector $\bar{x}$ that minimizes the length of $A\bar{x} - g$ with respect to our new inner product. As before, the length of $A\bar{x} - g$ is minimized when $A\bar{x} - g$ is perpendicular to the column space of A. It is easy to see that $A\bar{x} - g$ is perpendicular to the column space of A with respect to the inner product induced by Q if and only if it satisfies the equation $A^tQ(A\bar{x}-g)=0$, or, equivalently, $A^tQA\bar{x}=A^tQg$. If the columns of A are independent, then A^tQA is invertible, and we arrive at our solution:

$$\bar{x}=\left(A^tQA\right)^{-1}A^tQg.$$

Now we can write the solution out in terms of our inner product:

$$\bar{x}=\begin{pmatrix}\langle c,c\rangle & \langle c,h\rangle\\ \langle c,h\rangle & \langle h,h\rangle\end{pmatrix}^{-1}\begin{pmatrix}\langle c,g\rangle\\ \langle h,g\rangle\end{pmatrix}=\begin{pmatrix}\dfrac{\langle h,h\rangle\langle c,g\rangle-\langle c,h\rangle\langle h,g\rangle}{\langle c,c\rangle\langle h,h\rangle-\langle c,h\rangle^2}\\ \dfrac{\langle c,c\rangle\langle h,g\rangle-\langle c,h\rangle\langle c,g\rangle}{\langle c,c\rangle\langle h,h\rangle-\langle c,h\rangle^2}\end{pmatrix}.$$

Note that this is exactly the same solution as before except that in place of the dot product we have the inner product $\langle,\rangle$ given by the formula

$$\langle v,w\rangle=\sum\frac{v_jw_j}{ac_j+bh_j}.$$

Note also that in the special case in which each time between filling the tank the car is driven only in the city or only on the highway, the above solution reduces to (4), as desired.

What values should be used for a and b? The optimal values depend on how the variations in gas mileage in the city and on the highway compare. If we assume that the two variations are the same, then one can show that we should take $a=x_1$ and $b=x_2$, but of course we can't actually do that since we are trying to *find* x_1 and x_2. We could use an iterative procedure, but there is really no need to do so because the result is not very sensitive to the values used. It is quite satisfactory simply to take $a=b=1$. (Note that only the *ratio* a/b matters.) For instance, as shown in Tables 1 and 2, the results obtained from the data there with $a=x_1$ and $b=x_2$ differ by less than 0.1 mi/gal from the results obtained with $a=b=1$.

It is clear that, at least in theory, the weighted method is superior to the unweighted method. In practice, the two methods yield similar results, although the values obtained using the unweighted method tend to be slightly too high. This is illustrated by the results in Table 1. Using the gas mileages computed by the unweighted method, the total amount of gas predicted to have been used is 480 gallons, whereas the total amount actually used is 487 gallons. Naturally this discrepancy does not occur with the weighted method. It is easy to understand why the results of the unweighted method tend to be too high: the unweighted method in effect gives too much weight to those intervals in which many miles are driven, and these tend to be the intervals with higher gas mileage since higher gas mileage allows more miles to be driven before refueling.

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