Computing the Fundamental Matrix for a Reducible Markov Chain

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1. Career opportunity at a consulting firm The primary purpose of this paper is to show new techniques for computing the fundamental matrix of a reducible Markov chain. We shall apply the techniques to a model of a career path at a consulting firm. Suppose a firm employs three types of consultants: probationary, associate, and partner. Probationary consultants are classified as grade 1, 2, 3, and 4. Entry-level is grade 1 probationary. Only probationary consultants can be promoted or discharged. Discharged employees are never rehired and consultants are never demoted. Associates are classified as associates in engineering, in computing, or in mathematics. All changes in job classification occur at the end of a year.

1. What is the expected length of time that a newly hired grade 1 consultant spends with the firm as a grade 2 consultant?
2. What is the expected length of time that a newly hired grade 1 consultant spends as a probationary consultant?
3. What is the probability that a grade 1 consultant will be promoted to partner?
4. In the long run what proportion of associate consultants will be associates in engineering?

To answer these questions we use the standard model: the career path of a consultant as a Markov chain.

2. Markov chain model A Markov chain [2], [5] is a collection of random variables $\{X_t\}$ where the index t runs through the nonnegative integers. A state X_t represents the job classification of an employee at the end of year t. The conditional probabilities $P(X_{t+1}=j|X_t=i)$ are called transition probabilities. They do not change over time, that is, they are independent of t and are denoted by p_{ij}.

Our Markov chain model has 9 states indexed as follows.

9: discharged, 8: partner
7, 6, 5: associate in mathematics, computing, or engineering
4, 3, 2, 1: probationary consultant, grade 4, 3, 2, or 1,

which give rise to the transition matrix $P=[p_{ij}]$, $1 \le i, j \le 9$. As the p_{ij} are conditional probabilities, they must satisfy the properties

$$p_{ij} \ge 0, \quad \text{for all } i \text{ and } j.$$

$$\sum_{j=1}^{9} p_{ij} = 1, \quad \text{for all } i.$$

The directed graph in Figure 1 shows allowable transitions for the career of a consultant.

The states in our model are classified to describe the probabilistic behavior of employees. A state j is *accessible* from a state i if it is possible to enter j after one or more steps (time units), starting from i. In our model, states 5, 6, 7, 8, and 9 are accessible from states 1, 2, 3, and 4. If state j is accessible from state i and i is accessible from j, then states i and j are said to *communicate*. States 5 and 6 communicate. A set of states forms a *closed set* if no state outside the set is accessible from any state in the set. Once a process enters a closed set, it can never leave the set. If, in addition, each pair of states in a closed set communicate, the set is termed a *closed communicating class*. States 5, 6, and 7 form a closed communicating class. If a closed set contains only one state, that state is called an *absorbing state* and when the process enters an absorbing state, it can never leave. If i is an absorbing state, $p_{ii} = 1$. States 8 and 9 are absorbing states.

A Markov chain is *irreducible* if every state is accessible from every state. For an irreducible chain, a *steady state probability*, denoted by π_j, can be interpreted as the long run proportion of time that the process spends in state j. We can always find the values of π_j for an irreducible Markov chain ([**5**], pp. 151–152).

A nonirreducible Markov chain is termed *reducible*. Our consulting firm is modeled as a reducible Markov chain. In a reducible chain, any state that does not belong to a closed communicating class is called a *transient state*. If a Markov chain starts in a transient state, the chain is certain eventually to enter some closed communicating class ([**2**], p. 43). If a state is not transient, it is called *recurrent*. Starting from a recurrent state, eventual return to this state is certain. Our model has three classes of states: $\{8,9\}$ absorbing states, $\{5,6,7\}$ recurrent states, and $\{1,2,3,4\}$ transient states.

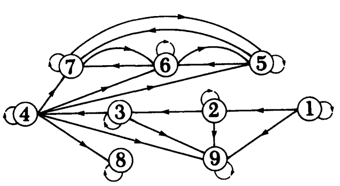

FIGURE 1

3. Canonical form of the transition matrix To answer our questions we arrange the transition matrix P in *canonical form*. Generally, states belonging to a communicating class are numbered consecutively. In a reducible chain with a closed communicating set of recurrent states and a set of transient states, we write P as the partitioned matrix

$$P = \begin{bmatrix} S & O \\ E & Q \end{bmatrix}, \tag{1}$$

where the square submatrix S is the transition matrix corresponding to the closed communicating class and the submatrix E specifies transitions from transient states to recurrent states. The submatrix Q governs transitions among the transient states.

4. State reduction The matrix construction algorithm is based on state reduction, an iterative procedure in which each iteration produces a reduced matrix one state smaller than its predecessor, resulting in a final reduced matrix from which the solution to the original problem can be obtained. Other state reduction algorithms are described in references [**1**], [**3**], and [**6**].

5. Probabilistic motivation The probabilistic motivation for state reduction is the following result in Kemeny and Snell ([**2**], pp. 114–115). Suppose that we have a Markov chain with an $n \times n$ transition matrix F partitioned as

$$F = \begin{array}{c c} & \begin{array}{cc} C & Y \end{array} \\ \begin{array}{c} C \\ Y \end{array} & \begin{bmatrix} Z & W \\ V & X \end{bmatrix} \end{array}. \tag{2}$$

Assume that we observe the process only when it is in a subset C of the states having c elements. A new Markov chain with c states, which we call a reduced process, is obtained. A single step in the reduced process corresponds in the original process to the transition, not necessarily in one step, from a state in C to another state in C. We compute the transition matrix D for the reduced process. Matrix Z describes transitions within C and has dimensions $c \times c$. Matrix W, with dimensions $c \times (n-c)$, governs transitions from C to Y, the subset of states outside of C. Matrix V describes transitions from Y to C and has dimensions $(n-c) \times c$. Matrix X, which governs transitions within Y, is $(n-c) \times (n-c)$. Let $Z = [z_{ij}]$, $W = [w_{ij}]$, $V = [v_{ij}]$, and $X = [x_{ij}]$. Let k and l be two states of C. We define

$$\begin{aligned} d_{kl} &= z_{kl} + \sum_{g,\, h \in Y} w_{kg}\left[x_{gh}^0 v_{hl} + x_{gh}^1 v_{hl} + x_{gh}^2 v_{hl} + \cdots\right] \\ &= z_{kl} + \sum_{g,\, h \in Y} w_{kg}\left[1 + x_{gh} + x_{gh}^2 + \cdots\right] v_{hl} \\ &= z_{kl} + \sum_{g,\, h \in Y} w_{kg}\left[1 - x_{gh}\right]^{-1} v_{hl} \quad \text{for } 1 \leqslant k, l \leqslant c. \end{aligned}$$

In matrix form we have

$$D = Z + W[I - X]^{-1} V \tag{3}$$

where I is the $(n-c) \times (n-c)$ identity matrix. Note D is $c \times c$.

6. Steady state probabilities for recurrent states Since our model of the consulting firm is a reducible Markov chain, we arrange the transition probability matrix P in canonical form.

$$P = \begin{array}{c c} & \begin{array}{ccccccccc} 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array} \\ \begin{array}{c} 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array} & \left[\begin{array}{c|c|ccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0.50 & 0.30 & 0.20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.30 & 0.45 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.10 & 0.35 & 0.55 & 0 & 0 & 0 & 0 \\ \hline 0.20 & 0.16 & 0.04 & 0.03 & 0.02 & 0.55 & 0 & 0 & 0 \\ 0.15 & 0 & 0 & 0 & 0 & 0.20 & 0.65 & 0 & 0 \\ 0.10 & 0 & 0 & 0 & 0 & 0 & 0.15 & 0.75 & 0 \\ 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0.10 & 0.85 \end{array}\right] \end{array}$$

The submatrix Q of transient states appears in the lower right-hand corner. The submatrix S of recurrent states represents an irreducible Markov chain.

$$Q = \begin{array}{c c} & \begin{array}{cccc} 4 & 3 & 2 & 1 \end{array} \\ \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} & \begin{bmatrix} 0.55 & 0 & 0 & 0 \\ 0.20 & 0.65 & 0 & 0 \\ 0 & 0.15 & 0.75 & 0 \\ 0 & 0 & 0.10 & 0.85 \end{bmatrix} \end{array},$$

$$S = \begin{array}{c|ccc|} & 7 & 6 & 5 \\ \hline 7 & 0.50 & 0.30 & 0.20 \\ 6 & 0.30 & 0.45 & 0.25 \\ 5 & 0.10 & 0.35 & 0.55 \end{array}.$$

To answer question 4 we solve the system of linear equations

$$\pi = \pi S,$$
$$1 = \pi_5 + \pi_6 + \pi_7,$$

to compute the steady state probability vector [5] for S,

$$\pi = \begin{array}{c} 7 \qquad 6 \qquad 5 \\ [\,0.291\,,\ 0.373\,,\ 0.336\,] \end{array}.$$

In the long run, 0.336 of the associate consultants will be in state 5, engineering.

7. Matrix construction algorithm The matrix construction algorithm we present is new. It contains two steps: *matrix augmentation* and *matrix reduction*. In the *matrix augmentation* step the transition matrix is truncated to obtain a submatrix Q governing transitions among the transient states. A null matrix and two identity matrices are adjoined to Q to form an augmented matrix B. In the *matrix reduction* routine, B is reduced to produce a matrix B_n that is the same size as Q.

In the matrix augmentation step, we assume that the transition matrix P for a reducible Markov chain is partitioned as in (1). For any reducible Markov chain the inverse of the matrix $(I - Q)$ exists and is called the *fundamental matrix*, N (see [2]). To construct an augmented matrix B of order $2n$ we assume that Q is $n \times n$. We adjoin a null matrix O, and two identity matrices I, each of order n, to Q. We arrange the augmented $2n \times 2n$ matrix in the form

$$B = \begin{bmatrix} O & I \\ I & Q \end{bmatrix}. \tag{4}$$

We let $B = [b_{ij}]$ and $Q = [q_{ij}]$.

The detailed steps of matrix reduction applied to the augmented matrix B are presented below.

1. Initialize $k = 2n$.
2. Let $B_k = B$.
3. Partition B_k as

$$\begin{array}{cc} & \begin{array}{cc} k-1 & \quad 1 \end{array} \\ B_k = & \left[\begin{array}{c|c} T_k & U_k \\ \hline R_k & Q_k \end{array}\right] \begin{array}{c} k-1 \\ 1 \end{array}. \end{array}$$

4. $B_{k-1} = T_k + U_k[I - Q_k]^{-1}R_k$ where $[I - Q_k]^{-1} = (1 - b_{kk})^{-1}$.
5. Decrement k by 1. If $k = n$, stop. Otherwise, repeat steps 3 and 4.

Matrix reduction ends when $k = n$, indicating that the final reduced matrix, B_n, is of order n. The calculation of the fundamental matrix is based on the following application of equation (3): When a matrix B is partitioned as in (4), then

$$B_n = O + I[I - Q]^{-1}I = [I - Q]^{-1} = N. \tag{5}$$

Therefore, B_n is the fundamental matrix.

8. Operation count Each step of matrix reduction performs $(n-1)^2$ additions, 1 subtraction, (n^2-1) multiplications, and 1 division. Multiplying these numbers by n steps, we have (n^3-2n^2+2n) additions and subtractions and n^3 multiplications and divisions. If we count only multiplications and divisions, then the number of operations is approximately n^3. A standard method for inverting a matrix such as the Crout LU decomposition also performs approximately n^3 operations [4].

9. Analysis of consulting opportunity By applying results derived by Kemeny and Snell for the fundamental matrix of a reducible Markov chain, we answer the first three questions.

$$B = \begin{array}{c} \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array}\begin{array}{c} \begin{array}{cccccccc} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array} \\ \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0.55 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.20 & 0.65 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.15 & 0.75 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0.10 & 0.85 \end{array}\right] \end{array} = \begin{bmatrix} O & I \\ I & Q \end{bmatrix}.$$

$$B_8 = \begin{array}{c} \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array}\begin{array}{c} \begin{array}{cccccccc} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0.55 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.20 & 0.65 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.15 & 0.75 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0.10 & 0.85 \end{array}\right] \end{array} = \begin{bmatrix} T_8 & U_8 \\ R_8 & Q_8 \end{bmatrix}.$$

$$B_7 = T_8 + U_8[I - Q_8]^{-1} R_8.$$

$$\vdots$$

$$B_4 = T_5 + U_5[I - Q_5]^{-1} R_5 = N = \begin{array}{c} \\ 4 \\ 3 \\ 2 \\ 1 \end{array}\begin{array}{c} \begin{array}{cccc} 4 & 3 & 2 & 1 \end{array} \\ \begin{bmatrix} 2.222 & 0 & 0 & 0 \\ 1.270 & 2.857 & 0 & 0 \\ 0.762 & 1.714 & 4 & 0 \\ 0.508 & 1.143 & 2.667 & 6.667 \end{bmatrix} \end{array}.$$

To answer question 1, if a process is presently in a transient state i, the expected number of periods that will be spent in a transient state j before entry into an absorbing or a recurrent state is the (i,j)th element of the fundamental matrix. Therefore, the expected time that a grade 1 consultant spends with the firm as a grade 2 consultant is 2.667 years, element $(1,2)$ of N.

To answer question 2, if a process is presently in a transient state i, the expected number of periods that will be spent in all transient states before entry into an absorbing or a recurrent state is the sum of the elements in the ith row of the fundamental matrix. The expected time that a grade 1 consultant spends as a probationary consultant is equal to the sum of the expected times that she spends in grades 1 through 4, or $6.667 + 2.667 + 1.143 + 0.508 = 10.985$ years.

To answer question 3, if a process is presently in a transient state i, the probability of eventual absorption in an absorbing state j is the (i,j)th element of the matrix NE_1, where E_1 is the submatrix governing transitions from transient states to absorbing states. We have

$$NE_1 = \begin{array}{c} \\ 4 \\ 3 \\ 2 \\ 1 \end{array}\begin{array}{c} \begin{array}{cccc} 4 & 3 & 2 & 1 \end{array} \\ \begin{bmatrix} 2.222 & 0 & 0 & 0 \\ 1.270 & 2.857 & 0 & 0 \\ 0.762 & 1.714 & 4 & 0 \\ 0.508 & 1.143 & 2.667 & 6.667 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{cc} 9 & 8 \end{array} \\ \begin{bmatrix} 0.20 & 0.16 \\ 0.15 & 0 \\ 0.10 & 0 \\ 0.05 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \\ 4 \\ 3 \\ 2 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 9 & 8 \end{array} \\ \begin{bmatrix} 0.444 & 0.356 \\ 0.683 & 0.203 \\ 0.810 & 0.122 \\ 0.873 & 0.081 \end{bmatrix} \end{array}.$$

The probability that a grade 1 consultant will be promoted to partner is 0.081, element $(1,8)$ of NE_1.

The matrix construction algorithm is an interesting new procedure for calculating the fundamental matrix for a reducible Markov chain.

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