NOTES

Raising the Roots

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Without knowing the roots of a given polynomial equation, how do you find the equation whose roots are some fixed power of the roots of the original?

This simple question can connect many topics from high school and undergraduate mathematics. This paper takes one path through the question, showing applications of ideas from algebra and linear algebra and previewing some more advanced topics.

Quadratic equations Let's start with degree 2, and suppose our quadratic equation is written as $x^2 - 2abx + b^2 = 0$. If its roots are α_1 and α_2, then

$$\alpha_1 + \alpha_2 = 2ab \quad \text{and} \quad \alpha_1 \alpha_2 = b^2.$$

An equation whose roots are α_1^k and α_2^k is

$$x^2 - \left(\alpha_1^k + \alpha_2^k\right)x + \alpha_1^k \alpha_2^k = 0$$

so the problem comes down to finding $\alpha_1^k + \alpha_2^k$ and $\alpha_1^k \alpha_2^k$ in terms of the coefficients of the original equation.

Well, the product is easy:

$$\alpha_1^k \alpha_2^k = (\alpha_1 \alpha_2)^k = b^{2k}.$$

So the object of the game is to express $\alpha_1^k + \alpha_2^k$ in terms of a and b. Let's look for a recursion.

$$\begin{aligned}\alpha_1^2 + \alpha_2^2 &= (\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2) - 2\alpha_1 \alpha_2 \\ &= (2ab)^2 - 2b^2 \\ &= 2b^2(2a^2 - 1)\end{aligned}$$

$$\begin{aligned}\alpha_1^3 + \alpha_2^3 &= (\alpha_1 + \alpha_2)\left(\alpha_1^2 + \alpha_2^2\right) - \alpha_1 \alpha_2 (\alpha_1 + \alpha_2) \\ &= 2ab\left(2b^2(2a^2 - 1)\right) - b^2(2ab) \\ &= 2b^3\left(2a(2a^2 - 1) - a\right) \\ &= 2b^3(4a^3 - 3a)\end{aligned}$$

$$\begin{aligned}\alpha_1^4 + \alpha_2^4 &= (\alpha_1 + \alpha_2)(\alpha_1^3 + \alpha_2^3) - \alpha_1\alpha_2(\alpha_1^2 + \alpha_2^2)\\ &= 2ab(2b^3(4a^3 - 3a)) - b^2(2b^2(2a^2 - 1))\\ &= 2b^4(2a(4a^3 - 3a) - (2a^2 - 1))\\ &= 2b^4(8a^4 - 8a^2 + 1)\\ &\vdots \qquad \vdots\end{aligned}$$

Inductively, we see that $\alpha_1^k + \alpha_2^k = 2b^k t_k(a)$, where $t_k(a)$ is a polynomial in a of degree k. In fact,

$$t_1(a) = a, \quad t_2(a) = 2a^2 - 1, \quad t_3(a) = 4a^3 - 3a, \quad \text{and} \quad t_4(a) = 8a^4 - 8a^2 + 1.$$

Furthermore, for $k > 2$, we have

$$\begin{aligned}2b^k t_k(a) &= \alpha_1^k + \alpha_2^k\\ &= (\alpha_1 + \alpha_2)(\alpha_1^{k-1} + \alpha_2^{k-1}) - \alpha_1\alpha_2(\alpha_1^{k-2} + \alpha_2^{k-2})\\ &= 2ab(2b^{k-1}t_{k-1}(a)) - b^2(2b^{k-2}t_{k-2}(a))\\ &= 2b^k(2at_{k-1}(a) - t_{k-2}(a)),\end{aligned}$$

so that $t_k(a) = 2at_{k-1}(a) - t_{k-2}(a)$, and we have a recursion for the t_k:

$$t_k(a) = \begin{cases} a & \text{if } k = 1;\\ 2a^2 - 1 & \text{if } k = 2;\\ 2at_{k-1}(a) - t_{k-2}(a) & \text{if } k > 2.\end{cases}$$

These are the *Chebyshev polynomials* (one usually starts with $t_0(a) = 1$, an equation that makes sense in the current context). Much is known about these polynomials; one of their many beautiful properties is that they yield trigonometric identities. This can be seen as follows:

Let $\theta \in \mathbb{R}$, $\alpha = \cos\theta + i\sin\theta$, and $\beta = \cos\theta - i\sin\theta$. Then α and β are roots of $x^2 - 2ax + 1$, where $a = \cos\theta$. So,

$$\begin{aligned}2t_k(a) &= \alpha^k + \beta^k\\ &= (\cos\theta + i\sin\theta)^k + (\cos\theta - i\sin\theta)^k\\ &= 2\cos k\theta.\end{aligned}$$

Since $a = \cos\theta$, we have $t_k(\cos\theta) = \cos k\theta$, so that the t_k provide a machine for generating the double, triple, quadruple, ... angle formulas for cosine:

$$\begin{aligned}\cos 2\theta = t_2(\cos\theta) &= 2\cos^2\theta - 1\\ \cos 3\theta = t_3(\cos\theta) &= 4\cos^3\theta - 3\cos\theta\\ \cos 4\theta = t_3(\cos\theta) &= 8\cos^4\theta - 8\cos^2\theta + 1\\ \vdots \qquad & \vdots\end{aligned}$$

Higher degrees Next, we let the degree increase and keep k constant. Suppose $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = f(x)$ has roots $\{\alpha_1, \ldots, \alpha_n\}$. How do you find an equation whose roots are the α_i^k?

The situation is more complicated here, because there's more to worry about than the sum and the product of the roots. But the question should be answerable. Since

$$f(x) = a_n(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n), \tag{1}$$

the coefficients of f are the *elementary symmetric functions* of the roots:

$$\alpha_1 + \alpha_2 \cdots + \alpha_n = -\frac{a_{n-1}}{a_n}$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 \cdots + \alpha_{n-1}\alpha_n = \frac{a_{n-2}}{a_n}$$

$$\sum_{h<i<j} \alpha_h\alpha_i\alpha_j = -\frac{a_{n-3}}{a_n}$$

$$\vdots \qquad \vdots$$

$$\alpha_1\alpha_2 \cdots \alpha_n = (-1)^n\frac{a_0}{a_n}$$

and to find an equation satisfied by the α_i^k, we'd have to express these symmetric functions of the α_i^k in terms of the a_i. A simple one is the constant term:

$$\alpha_1^k\alpha_2^k \dots \alpha_n^k = \left((-1)^n\frac{a_0}{a_n}\right)^k.$$

Newton gave a formula for the sum of the k^{th} powers of the roots in terms of the a_i. And the *theorem on symmetric functions* says that *any* symmetric function of the roots (including the ones we care about) can be expressed as a polynomial in the a_i (for details, see **[1]**). The expressions can get quite complex. It turns out that there is another approach that allows the calculation of the equation whose roots are powers of the roots of a given equation without recourse to symmetric functions. It works like this:

Suppose we want to find the equation satisfied by the squares of the roots of $f(x) = 0$. Replace x by $-x$ in (1):

$$f(-x) = a_n(-x-\alpha_1)(-x-\alpha_2)\dots(-x-\alpha_n). \tag{2}$$

Multiply this together with (1):

$$f(x)f(-x) = \pm a_n^2(x^2-\alpha_1^2)(x^2-\alpha_2^2)\dots(x^2-\alpha_n^2). \tag{3}$$

But $f(x)f(-x)$ is a polynomial in x^2, say $g(x^2)$. Now replace x^2 by y on both sides of (3), and you get

$$g(y) = \pm a_n^2(y-\alpha_1^2)(y-\alpha_2^2)\dots(y-\alpha_n^2).$$

So $g(y) = 0$ is an equation whose roots are the α_i^2.

For the cubes of the roots, we use a similar process:

Suppose, as before, that $f(x) = a_0 + a_1x + \cdots + a_nx^n$ has zeros $\alpha_1, \dots, \alpha_n$. To simplify the calculations, group the terms of f by powers of 3, and let

$$A = a_0 + a_3x^3 + a_6x^6 + \cdots; \quad B = a_1 + a_4x^3 + a_7x^6 + \cdots;$$
$$C = a_2 + a_5x^3 + a_8x^6 + \cdots.$$

Then $f(x) = A + Bx + Cx^2$. Now let $\omega = e^{2\pi i/3}$. Replace x by ωx and $\omega^2 x$ to get three equations:

$$f(x) = A + Bx + Cx^2 = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n); \tag{4}$$

$$f(\omega x) = A + B\omega x + C\omega^2x^2 = (\omega x-\alpha_1)(\omega x-\alpha_2)\dots(\omega x-\alpha_n); \tag{5}$$

$$f(\omega^2 x) = A + B\omega^2 x + C\omega x^2 = (\omega^2 x-\alpha_1)(\omega^2 x-\alpha_2)\dots(\omega^2 x-\alpha_n). \tag{6}$$

I've used here the fact that A, B, and C are invariant under the substitutions (because they are polynomials in x^3). Multiply (4), (5), and (6) together to get:

$$\begin{aligned} f(x)f(\omega x)f(\omega^2 x) &= A^3 + (B^3 - 3ABC)x^3 + C^3x^6 \\ &= \pm(x^3 - \alpha_1^3)(x^3 - \alpha_2^3)\dots(x^3 - \alpha_n^3). \end{aligned} \tag{7}$$

This is a polynomial in x^3, so you can substitute y for x^3 on both sides, and you have a polynomial whose roots are the α_i^3.

EXAMPLE. Suppose $f(x) = -156 + 124x - 75x^2 + 35x^3 - 9x^4 + x^5$. The roots of f are $\{3, \pm 2i, 3 \pm 2i\}$. To get the equation whose roots are the squares of these, we want

$$f(x)f(-x) = 24336 + 8024x^2 - 247x^4 - 123x^6 + 11x^8 - x^{10}.$$

Put $y = x^2$ to get a fifth-degree polynomial whose roots are the squares of the roots of f (as can be checked by direct substitution):

$$24336 + 8024y - 247y^2 - 123y^3 + 11y^4 - y^5.$$

For the cubes, write f as $f(x) = A + Bx + Cx^2$, where

$$A = -156 + 35x^3, \quad B = 124 - 9x^3, \quad \text{and} \quad C = -75 + x^3.$$

Then, our "norm equation" (7) says that

$$f(x)f(\omega x)f(\omega^2 x) = A^3 + (B^3 - 3ABC)x^3 + C^3x^6,$$

which simplifies to

$$-3796416 + 109504x^3 - 59895x^6 + 1775x^9 - 9x^{12} + x^{15}.$$

Putting $y = x^3$, we get a polynomial whose roots are the cubes of the roots of f (a fact one can check by direct substitution):

$$-3796416 + 109504y - 59895y^2 + 1775y^3 - 9y^4 + y^5.$$

Doing it in general To form the equation whose roots are the kth powers of the roots of f, we'd have to form the product of all the "conjugates" of f:

$$f(x)\cdot f(\zeta x)\cdot f(\zeta^2 x)\cdot f(\zeta^3 x)\cdots f(\zeta^{k-1}x),$$

where $\zeta = \cos\frac{2\pi}{k} + i\sin\frac{2\pi}{k}$. This process of forming all the conjugates of a polynomial and multiplying these together leads to messy calculations with complex numbers. Fortunately, there's more classical mathematics that can help out, allowing us to work entirely with polynomials over the original coefficient ring.

A useful idea in algebra is to let an element of a system "act on" the system, looking at the element as both a member of the system and a function on the system and using the algebra in the system to define function application. Understanding this action often leads to an understanding of the actor. In our case, suppose

$$f(x) = A_0 + A_1x + A_2x^2 + \cdots + A_{k-1}x^{k-1},$$

where the $A_i = A_i(x^k)$ are polynomials in x^k, invariant under the substitutions we are about to make, and consider its "first" conjugate:

$$f(\zeta x) = A_0 + A_1\zeta x + A_2\zeta^2x^2 + \cdots + A_{k-1}\zeta^{k-1}x^{k-1}.$$

Call this thing β_1. Let's look at the effect of β_1 on the powers of ζ:

$$\begin{aligned}
\beta_1 \cdot 1 &= A_0 + A_1 \zeta x + A_2 \zeta^2 x^2 + \cdots + A_{k-1} \zeta^{k-1} x^{k-1} \\
\beta_1 \cdot \zeta &= A_{k-1} x^{k-1} + A_0 \zeta + A_1 \zeta^2 x + A_2 \zeta^3 x^2 + \cdots + A_{k-2} \zeta^{k-1} x^{k-2} \\
\beta_1 \cdot \zeta^2 &= A_{k-2} x^{k-2} + A_{k-1} \zeta x^{k-1} + A_0 \zeta^2 + A_1 \zeta^3 x + \cdots + A_{k-3} \zeta^{k-1} x^{k-3} \\
\vdots &= \vdots \ \ddots \\
\beta_1 \cdot \zeta^{k-1} &= A_1 x + A_2 \zeta x^2 + A_3 \zeta^2 x^3 + \cdots + A_0 \zeta^{k-1}.
\end{aligned}$$

Write this as a matrix equation:

$$\beta_1 \begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \\ \vdots \\ \zeta^{k-1} \end{pmatrix} = \begin{pmatrix} A_0 & A_1 x & A_2 x^2 & A_3 x^3 & \cdots & A_{k-1} x^{k-1} \\ A_{k-1} x^{k-1} & A_0 & A_1 x & A_2 x^2 & \cdots & A_{k-2} x^{k-2} \\ A_{k-2} x^{k-2} & A_{k-1} x^{k-1} & A_0 & A_1 x & \cdots & A_{k-3} x^{k-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1 x & A_2 x^2 & A_3 x^3 & A_4 x^4 & \cdots & A_0 \end{pmatrix} \begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \\ \vdots \\ \zeta^{k-1} \end{pmatrix}.$$

Subtracting gives

$$\begin{pmatrix} A_0 - \beta_1 & A_1 x & A_2 x^2 & A_3 x^3 & \cdots & A_{k-1} x^{k-1} \\ A_{k-1} x^{k-1} & A_0 - \beta_1 & A_1 x & A_2 x^2 & \cdots & A_{k-2} x^{k-2} \\ A_{k-2} x^{k-2} & A_{k-1} x^{k-1} & A_0 - \beta_1 & A_1 x & \cdots & A_{k-3} x^{k-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1 x & A_2 x^2 & A_3 x^3 & A_4 x^4 & \cdots & A_0 - \beta_1 \end{pmatrix} \begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \\ \vdots \\ \zeta^{k-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Because this equation takes place in an integral domain, the determinant of the left-hand matrix is zero.

Similarly, if $0 \leq j \leq k-1$, and

$$\beta_j = f(\zeta^j x) = A_0 + A_1 \zeta^j x + A_2 \zeta^{2j} x^2 + \cdots + A_{k-1} \zeta^{(k-1)j} x^{k-1},$$

then

$$\beta_j \begin{pmatrix} 1 \\ \zeta^j \\ \zeta^{2j} \\ \zeta^{3j} \\ \vdots \\ \zeta^{(k-1)j} \end{pmatrix} = \begin{pmatrix} A_0 & A_1 x & A_2 x^2 & A_3 x^3 & \cdots & A_{k-1} x^{k-1} \\ A_{k-1} x^{k-1} & A_0 & A_1 x & A_2 x^2 & \cdots & A_{k-2} x^{k-2} \\ A_{k-2} x^{k-2} & A_{k-1} x^{k-1} & A_0 & A_1 x & \cdots & A_{k-3} x^{k-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1 x & A_2 x^2 & A_3 x^3 & A_4 x^4 & \cdots & A_0 \end{pmatrix} \begin{pmatrix} 1 \\ \zeta^j \\ \zeta^{2j} \\ \zeta^{3j} \\ \vdots \\ \zeta^{(k-1)j} \end{pmatrix}.$$

So

$$\begin{vmatrix} A_0-\beta_j & A_1x & A_2x^2 & A_3x^3 & \cdots & A_{k-1}x^{k-1} \\ A_{k-1}x^{k-1} & A_0-\beta_j & A_1x & A_2x^2 & \cdots & A_{k-2}x^{k-2} \\ A_{k-2}x^{k-2} & A_{k-1}x^{k-1} & A_0-\beta_j & A_1x & \cdots & A_{k-3}x^{k-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1x & A_2x^2 & A_3x^3 & A_4x^4 & \cdots & A_0-\beta_j \end{vmatrix} = 0$$

and $\{\beta_0,\dots,\beta_{k-1}\}$ (that is, the conjugates of f) are all roots of the following polynomial equation in t:

$$\begin{vmatrix} A_0-t & A_1x & A_2x^2 & A_3x^3 & \cdots & A_{k-1}x^{k-1} \\ A_{k-1}x^{k-1} & A_0-t & A_1x & A_2x^2 & \cdots & A_{k-2}x^{k-2} \\ A_{k-2}x^{k-2} & A_{k-1}x^{k-1} & A_0-t & A_1x & \cdots & A_{k-3}x^{k-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1x & A_2x^2 & A_3x^3 & A_4x^4 & \cdots & A_0-t \end{vmatrix} = 0. \qquad (8)$$

This equation is a polynomial in t with coefficients that are polynomials in x. Its roots are the conjugates of f. But we want the *product* of these conjugates. So, we want the constant term of the left side of (8) (up to a sign). But you get the constant term by putting $t=0$. In other words, the product of f and all its conjugates is the following "circulant" (see [2] and [3] for more on circulants):

$$\begin{vmatrix} A_0 & A_1x & A_2x^2 & A_3x^3 & \cdots & A_{k-1}x^{k-1} \\ A_{k-1}x^{k-1} & A_0 & A_1x & A_2x^2 & \cdots & A_{k-2}x^{k-2} \\ A_{k-2}x^{k-2} & A_{k-1}x^{k-1} & A_0 & A_1x & \cdots & A_{k-3}x^{k-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1x & A_2x^2 & A_3x^3 & A_4x^4 & \cdots & A_0 \end{vmatrix}$$

This will be a polynomial in x^k (a nice exercise), and putting $y=x^k$ produces a polynomial whose roots are the kth powers of the roots of f.

EXAMPLE. If $f(x)=A+Bx+Cx^2$ (where A, B, and C are polynomials in x^3), then the product of the conjugates of f is

$$\begin{vmatrix} A & Bx & Cx^2 \\ Cx^2 & A & Bx \\ Bx & Cx^2 & A \end{vmatrix} = A^3+(B^3-3ABC)x^3+C^3x^6$$

as before.

For the fourth powers of the roots, write f as $f(x)=A+Bx+Cx^2+Dx^3$, where the coefficients are polynomials in x^4. The desired polynomial can be obtained from the determinant:

$$\begin{vmatrix} A & Bx & Cx^2 & Dx^3 \\ Dx^3 & A & Bx & Cx^2 \\ Cx^2 & Dx^3 & A & Bx \\ Bx & Cx^2 & Dx^3 & A \end{vmatrix} = A^4+(-B^4+4AB^2C-2A^2C^2-4A^2BD)x^4$$

$$+(C^4-4BC^2D+2B^2D^2+4ACD^2)x^8-D^4x^{12}$$

by replacing x^4 by y. ■

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Invariants Under Group Actions to Amaze Your Friends

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Introduction This paper presents some simple magic tricks that work by themselves based upon mathematical principles. There are whole books devoted to this subject (see, e.g., [**2**], [**3**], or [**5**]), but this article specifically explores card effects that exploit invariants under the (group) action of mixing the cards. Brent Morris' recent book [**4**] is a wonderful exposition of the mathematics behind some of the effects that take advantage of the groups generated by perfect shuffles. This article orthogonally explores tricks where a *spectator* is allowed to mix the cards. The underlying theme is that a large permutation group leaves an audience with the feeling that the cards are being mixed while leaving an interesting set of invariants under the group action that can be used to perform a (hopefully) startling effect.

TV magic This trick was performed on a recent television program: A volunteer from the audience was handed the four aces from a deck of cards while the performer turned his back. (The reader may wish to take four playing cards, one from each suit, and follow along.) The magician then gave the following instructions:

1. Stack the four cards face-up with the heart at the bottom, then the club, then the diamond, and finally the spade.
2. Turn the spade (the uppermost card) face down.
3. Perform any of the following operations as many times and in any order that you wish:
   (a) Cut any number of cards from the top to the bottom.
   (b) Turn the top two cards over as one.
   (c) Either turn the entire stack over or do not—your choice.
4. Turn the topmost card over, then turn the top two cards over as one, and then turn the top three cards over as one.

At this point, the prestidigitator correctly divines that the club is the only card facing the opposite way from the others. As long as the audience member correctly followed the above directions, the magician is sure to be right.