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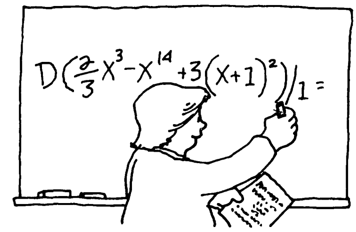
CLASSROOM CAPSULES



A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Nazanin Azarnia.

Calculus Measures Tank Capacity and Avoids Oil Spills

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Students find calculus abstract not because of a dearth of applications—textbooks contain thousands of them—but because of the lack of *documentation*, such as the case study provided below. For example, consider the following problem.

Problem 1. Imagine a fuel tank in the shape of a horizontal cylinder, with length l and circular cross section with diameter d. Moreover, suppose that fuel partially fills the tank, from the bottom to a height (depth) h above the bottom. Find a formula for the volume of fuel in the tank, denoted by $V(l, d, h)$.

The problem just stated may seem like any other allegedly dull calculus problem to practical students and instructors, who may also want to know:

Who needs the formula?
Why does anybody need the formula?
What would happen without the formula?

Answers to such questions rarely accompany textbook problems, which may seem dull but nevertheless apply to reality. In the following sections, I will document one application of problem 1, to convince students of the applicability of calculus.

Documentation. American Transport, Inc. (a firm based in Portland, Oregon), delivers fuel oils by trucks to underground tanks at customers' plants. Upon delivery, the truck driver first lowers a vertical graduated ruler—like a car's dip stick—into the tank to measure the amount of fuel in the tank and the remaining space available for delivery. This precaution will enable the driver to stop the truck's pump before overfilling the underground tank and spilling oil. The driver needs a means to convert the reading of depth on the vertical ruler into an estimate of the volume of fuel in the tank, such as a conversion chart. Unfortunately, such charts cannot be readily produced through experiments at each new customer's tank. Is there then a formula to generate charts by computer? This was, in essence, the telephone inquiry from American Transport's office in Spokane, Washington, on 3 January 1991: Solve problem 1!

Observe that the problem neither involves nor asks for any number; instead, it asks for a literal formula. Also, notice that the problem does not come with a picture, partly because it came via telephone, and partly because everyone in the oil business knows what a cylindrical tank looks like.

Solution. The following solution shows what skills students need to solve applied calculus problems. First, a picture may help intuition, and sketching one by hand may take less time than with a computer (see Figure 1). Second, translating the problem from its geometric statement into a calculus exercise cannot yet readily be done by a computer, and, consequently, must still be done mentally. To this end, identify the cross section of the tank with the disk of diameter d and center at the origin, characterized by the inequality $x^2 + z^2 \leq (d/2)^2$. Thus, the fuel in the tank occupies the region bounded by the inequalities (see Figures 1 and 2)

$$\begin{cases} -l/2 \leq y \leq l/2 & \text{(for a total length } l\text{)} \\ -d/2 \leq z \leq -d/2 + h & \text{(depth } z \text{ from bottom),} \\ \sqrt{(d/2)^2 - z^2} \leq x \leq \sqrt{(d/2)^2 - z^2} & \text{(width } x \text{ between the tank's curved sides at depth } z\text{).} \end{cases}$$

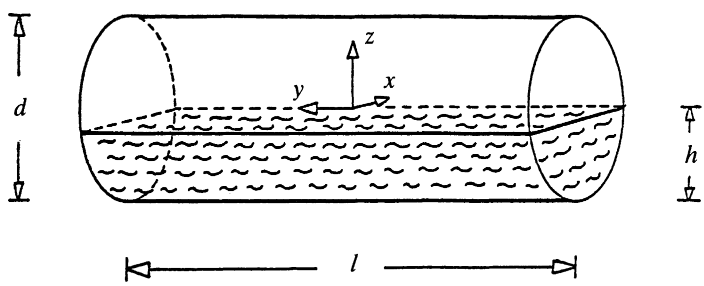

Figure 1
Calculate the volume of the fuel in terms of l, d, and h.

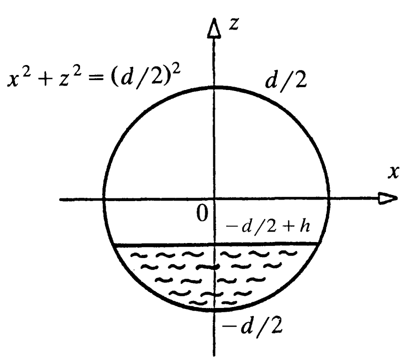

Figure 2
$-d/2 \leq z \leq -d/2 + h$ and $-\sqrt{(d/2)^2 - z^2} \leq x \leq \sqrt{(d/2)^2 - z^2}$.

Thus, the fuel's volume takes the integral form

$$V(l,d,h)=\int_{y=-l/2}^{y=l/2}\int_{z=-d/2}^{z=-d/2+h}\int_{x=-\sqrt{(d/2)^2-z^2}}^{x=\sqrt{(d/2)^2-z^2}}dx\,dz\,dy. \tag{0}$$

A routine calculation yields

$$V(l,d,h)=l\cdot\left\{\left(\frac{d}{2}\right)^2\cdot\left(\frac{\pi}{2}-\operatorname{Arcsin}\left(1-\frac{2h}{d}\right)\right)+\left(h-\frac{d}{2}\right)\cdot\sqrt{h(d-h)}\right\}. \tag{1}$$

The identity $(\pi/2)-\operatorname{Arcsin} t=\operatorname{Arccos} t$, inverse to the trigonometric identity $\cos(\pi/2-\alpha)=\sin\alpha$, leads to the simpler formula

$$V(l,d,h)=l\cdot\left\{\left(\frac{d}{2}\right)^2\cdot\operatorname{Arccos}\left(1-\frac{2h}{d}\right)+\left(h-\frac{d}{2}\right)\cdot\sqrt{h(d-h)}\right\}. \tag{2}$$

Verification. Because of the importance of formula (2) in practice, as already explained, and because of the unpleasant consequences that an error may cause, several verifications appear appropriate. Such verifications form the object of the following problems, which, to simulate reality, appear here without answers.

Problem 2: Alternate derivation. Arrive at formula (2) in a different manner, by integrating not the areas of the vertical cross sections, but the areas of the rectangular horizontal slices of the fuel. What multiple integral now gives the volume? What theorem about multiple integrals do the result of the present problem and formula (2) illustrate?

Problem 3: Geometric tests. Select a few values of h for which you can calculate the volume $V(l,d,h)$ geometrically, for example, $h=0$ (empty tank), $h=d/2$ (half-empty tank), and $h=d$ (full tank). Compare the expressions of $V(l,d,h)$ obtained through geometry with those given by formula (2).

Problem 4: Numerical tests. American Transport has requested sample values over the telephone to verify their implementation of formula (2) on their computer. Program formula (2) on a programmable calculator or computer. Consider a real tank 395 inches long and 95 inches in diameter. Verify with your calculator that such a tank has a capacity of about 12,000 gallons. Also, how many gallons of fuel should American Transport's computer indicate for a height of one inch at the bottom of the same tank?

Problem 5: Trigonometric variants. Some versions of `BASIC` offer neither the arccosine nor arcsine, but only the arctangent, abbreviated `ATN` (see [**2**]). To accommodate such compilers, rewrite formula (2) with an expression involving Arctan rather than Arccos, by means of trigonometric identities similar to those used to derive formula (2) from formula (1). Some symbolic computation programs, such as *Mathematica*, produce a formula for the iterated integral (0) that involves the Arctangent function [**5**].

Extension. The project described here consists of four problems, which address further inquiries from American Transport at increasing levels of difficulty.

On 17 August 1991, American Transport wrote to request a formula similar to (2) but for "fiberglass tanks that are shaped like a giant Tylenol capsule," which forms the object of the following problem.

Problem 6: Hemispherical ends (routine). Establish a formula, $T(l, d, h)$, for the volume of fuel with depth h at the bottom of a horizontal cylinder with length l, circular cross section of diameter d, and with *hemispherical caps* at both ends. In other words, the cylindrical tank considered here does not end with a vertical planar disc, but with a half of a sphere at each end. Thus, the *total* length of the tank is the sum $l + d$ of the length of the cylindrical part, l, and the diameter d, because the two hemispheres each add a radius, $d/2$, to the total length. Then test your formula with the dimensions of a real tank of this type, with a length of $l = 329$ inches and a diameter of $d = 96$ inches. Your formula should yield a volume $T(329, 96, 96)$ of *about* 12,000 gallons for the full tank.

On 13 February 1991, American Transport called again, requesting assistance in calculating the volume of fuel in tanks with "oval" cross sections.

Problem 7: Elliptical section (routine). Establish a formula, $W(l, a, b, h)$, for the volume of fuel with depth h at the bottom of a horizontal cylinder with length l and elliptical cross section with horizontal diameter a and vertical diameter b. The ends of the tank are vertical flat elliptical panels. Test your result with the dimensions of a real tank, in inches: $l = 172$, $a = 90$, $b = 63$, and $h = 63$ (full tank).

Problem 8: Ellipsoidal ends (moderate). In the preceding problem, suppose the ends of the tank are not vertical flat walls, but ellipsoidal caps with a bulge of three inches at the center. Thus, the total length of the tank along its axis is $3 + 172 + 3 = 178$ inches at the center, but still 172 inches along the perimeter. Find the volume of the tank.

Problem 9: "Oval" section (approximation). For a more accurate approximation of reality, take into account the fact that the cross section is not exactly elliptical. For the real tank in problem 7, the perimeter of the cross section is an "oval," symmetric with respect to the origin, and passing through the points in the first quadrant with coordinates $(45, 0)$, $(40, 15.75)$, $(22.5, 28)$, and $(0, 31.5)$.

Variations. For a greater challenge, students may try Strang's version of problem 1: The tank is neither horizontal nor vertical, but tilted [**4**]. Documentation for such a challenging calculus problem appears in Gray's note [**3**], where the design of a gauge for a storage tank in a chemical processing plant requires the calculation of the volume of water inside a tilted cylindrical pipe.

Alternatively, students may read Balk's account [**1**] of independent calculations, first by the Venerable Cooper, and later by Kepler in 1615, to measure the *full* capacity of Linz's wine barrels by dipping a stick from the middle of the side diagonally to the opposite internal "corner" of the barrel.

Conclusion. The case study presented here, thanks to American Transport, may contribute to a livelier calculus lecture. Yet the case study also confirms that the skills developed in calculus texts—be they lean and lively or mean and deadly—correspond to the skills required by real applications: reading and understanding problems stated in prose, drawing figures, translating prose into mathematics, setting up multiple integrals, using algebraic or trigonometric identities, and so forth. Furthermore, the case study shows that while calculus problems may seem dull or abstract without documentation by case studies, their applications may nevertheless be quite slick.

References

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Disks, Shells, and Integrals of Inverse Functions

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Most calculus students (and many calculus textbooks) take it for granted that the shell and disk methods for computing the volume of a solid of revolution must always give the same result. There are really two problems here: to show that the two methods agree, and that the result is the volume of the solid.

Once one has defined volume to be the result of integrating the constant function 1 over solid regions, it is easy to show that the shell method gives the volume, by using cylindrical coordinates. It remains to be shown that the disk method gives the same result. Several elementary proofs that the two methods agree have been published. In 1955, Parker [**4**] indicated a proof based on a simple relationship between the integrals of a function and its inverse. Later, Cable [**1**] gave a short proof using integration by parts. Both of these authors assumed that the function whose graph is revolved to form the boundary of the solid is monotone and has a continuous derivative. Recently Carlip [**2**] gave a proof that the disk and shell methods agree, assuming only continuity and monotonicity, based on the fundamental theorem of calculus and the principle that two functions with the same derivative that agree at a point are equal. In this note we show that Parker's original proof actually requires only these weaker hypotheses too.

Referring to Figure 1, we would expect that

$$\int_a^b h(x)\,dx + \int_{h(a)}^{h(b)} h^{-1}(y)\,dy = bh(b) - ah(a).$$

This relationship is the basis for a geometric interpretation of integration by parts, that occurs in Courant [**3**] and many other calculus texts. We sketch the formal proof below. I tell my students this is a "happy theorem" because we are able to formally prove a result that had better be true if our definition of integral successfully captures the notion of area.

Theorem 1. *Suppose that* $h\colon [a,b] \to [h(a), h(b)]$ *is monotone and continuous. Then* $\int_{h(a)}^{h(b)} h^{-1}(y)\,dy = bh(b) - ah(a) - \int_a^b h(x)\,dx$.